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THE USE OF LANCHESTER-TYPE EQUATIONS IN THE ANALYSIS OF PAST MILITARY ENGAGEMENTS

A THESIS

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THE USE OF LANCHESTER-TYPE EQUATIONS

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FOREWORD

The author's interest in military operations analysis stems from over 20 years service with the military, either on active duty or as a civil servant assigned to a military detachment. The author is indebted to Dr. John J. Murphy, Office of Technical Analysis, Air Force Armament Center, and Mr. Howard Dimmig of the Deputy for Effectiveness Test, Air Proving Ground Center, Eglin Air Force Base, Florida. Without their help in obtaining one full year of graduate training, the program leading to this research would not have been possible.

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To my wife Fern, son Eric, and daughter Anita, without whose understanding and encouragement this program could never have been completed, I dedicate this work.

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SUMMARY

Using Bodart's *Lexicon* (covering 1,081 land battles fought between 1620 and 1905) as a data source, an analysis of past military engagements was conducted with the following specific objectives:

(1) To determine if Lanchester's square law, Lanchester's linear law or some other exponential law best depicts the flow of combat.

(2) To develop empirical relationships between the effectiveness ratio and the initial strength ratio.

(3) To develop an advantage parameter to estimate the side possessing the advantage.

(4) To determine the sensitivity of the form of Lanchester's law to the magnitude of the effectiveness coefficients.

(5) To categorize military combat situations according to total force and per cent casualties and develop models to estimate effectiveness ratio and advantage for each category.

(6) To estimate the validity of the models developed by determining the stability of their regression coefficients with time.

Bivariate regression analysis techniques were used to analyze past military data and to develop the models for estimating effectiveness ratio and advantage. Both a graphical and a statistical analysis were used to analyze the stability of the models with time.

Based upon the results of this study, the following conclusions were obtained:

(1) In general, the battle situations studied were insensitive to the form of Lanchester's law used.

(2) Empirical relationships were established between the effectiveness ratio and the initial strength ratio.

(3) The advantage parameter developed was in good agreement with the winner of the battles studied.

(4) A sensitivity analysis, using simulation results supported conclusion one.

(5) A two-way classification of battles by battle size and per cent casualties showed that the classification by battle size did not improve the models' ability to estimate advantage. Classification by per cent casualties resulted in a 7 per cent improvement for those classes in which one side had heavy losses and the other light losses.

(6) Analyses of battles in which one side had light losses and the other heavy revealed that, on the average, the larger force has the larger unit effectiveness when the initial strength ratio is less than three to one. The reverse is true for larger ratios of initial strength.

(7) The models developed proved to be time invariant over the time period of the data.

(8) The distribution of battles by battle size follows an exponential distribution.

As a result of the literature search and problems encountered in this study, the following areas are recommended for further research:

(1) More research needs to be conducted into the nature of the effectiveness ratio with particular emphasis on such attributes as types

of weapons employed, types of terrain, communications, and logistics.

(2) The influence of such intangibles as morale of troops on the effectiveness ratio needs further study. Both Rashevsky and Weiss have made theoretical studies in this area, but no attempt has been made to validate these studies or to develop a new theory based upon analyses of past military conflicts.

(3) A final recommended area for research is a study of past military engagements directed toward the determination of an empirical relationship between the motion of the forward edge of the battle area (FEBA) and the effectiveness ratio.

CHAPTER I

A CRITICAL SURVEY OF EXISTING INFORMATION RELATING TO THE PRACTICE AND THEORY OF WARFARE

Purpose

Since a nation's very existence may depend on the outcome of its military engagements, men throughout the centuries have been interested in the problem of predicting the outcome of these engagements and determining the best strategies to follow in combat.

Military strategists have developed numerous varieties of war games and map maneuvers in an attempt to solve this problem. However, the first mathematical analysis of the relationships between opposing forces in battle was made by F. W. Lanchester (38) in 1916.

The general purpose of this research is to summarize past achievements and through new formulations to attempt to extend the practical use of Lanchester-type equations. Specific study objectives are stated at the end of this chapter.

Background

History of Warfare

War, as defined by Quincy Wright (62), is a violent contact between distinct but similar entities. Under this definition, an automobile wreck, an animal fight, a clash between two primitive tribes, or a conflict between two nations using the most modern equipment of destruction would all be classified as war. Karl von Clausewitz (14) defines war as "an act of force to compel our adversary to do our will." This definition, while more restrictive than Quincy Wright's, is still too broad in scope. For the purpose of this study, war shall be defined as "an act of violence by which disputes between governments are settled."

The history of war can be divided into three broad epochs: war between primitive men, war between civilized men, and modern war between large nations employing modern weapons. Quincy Wright points out that speech, writing, and printing, respectively, initiated the ages of primitive man, of civilization, and of the world community.

War can be traced to the emergence of the primates, able to communicate with each other by spoken language, many centuries ago. Communication is necessary to war, since it has been defined as an act of violence between two governments and only with communication is government possible. War between these primitive creatures was fought for atonement, for revenge, for sport, for sexual prestige, for territorial conquest or defense, but mainly for manifesting and preserving the solidarity of the group or government. Today we call this last characteristic patriotism or nationalism. The importance of group solidarity or patriotism is emphasized by the fact that in many primitive cultures, clans composed of only a few hundred persons were continuously at war. These wars were necessary to preserve internal solidarity. Camilla H. Wedgewood (62) makes this clear: "The constant function of war is to strengthen the bonds of union between the individuals of the fighting community and make them increasingly conscious that they are members of a single unit."

The second epoch of war, war between civilized man, probably began with the formation of civilization along the valleys of the Nile and Euphrates, approximately 10,000 years ago. War in the strict sense of the definition adopted for this study began with this era. Not until this time in history was it possible to give a clear distinction between ruler and ruled, a clear conception of property, and to have a body of laws, distinct from the natural instincts, to regulate the social relationships existing between men. A written language permitting the storage of ideas and records and communications between two separated parties was essential. Warfare during this period had a tendency to involve larger groups, since the written language allowed unified orders and commands to be given to much larger groups than the spoken language. In addition, the written language led to inventions of instruments of warfare, to military transportation, and to new military formations or tactics. The purposes of war remained much the same as they were during the pre-civilized era. War was fought for the most part by specialized personnel and one of its main purposes continued to be that of maintaining the solidarity of the group. Civilizations usually expanded in territory and became more integrated internally, all as a result of warfare. This same warfare then destroyed the civilization through destructiveness, the exhaustion of resources, epidemic diseases, and other causes.

The invention of the printing press in the early fifteenth century marked the beginning of the age of invention--the age of modern warfare. The invention of guns and explosives, the improvement of means of transportation, the application of steam and gasoline power all contributed to a new era of destructive warfare. Until recent periods, man had no reliable method of releasing power stored by other than human or animal muscle for the purpose of striking an enemy. The inventions of firearms and chemicals for striking the enemy and steam, gas, and electric power for military movement, coupled with a reliable means of communication have transformed the character of military operations and made them war in the modern sense. These changes have made war more destructive, more likely to spread, and consequently of more general interest to all mankind. Men in all civilizations have spoken of the evils of warfare but war, with civilization, has continuously progressed in its efficiency, and mankind is forever devoting more time and money to implementing newer and more powerful implements of destruction.

The last few centuries have seen a continued interest in developing a theory of warfare, for if we cannot have a warless world we must understand the science or art of the conduct of war. Karl von Clausewitz (14) aptly states:

> If theory investigates the things that make up war, if it separates more distinctly that which at first sight seems confused, if it explains fully the properties of the means, if it shows their probable effects, if it clearly defines the nature of the ends in view, if it sheds the light of a deliberate, critical observation over the whole field of war--then it has achieved the main object of its task.

The problem of analyzing military conflicts to determine the best strategies to follow in the conduct of warfare, has become of increasing importance in modern times.

War Gaming

The specific origins of early studies regarding the analysis and the simulation of military conflict are unclear. Sayer (54) makes the statement that war gaming evolved from what might be designated as "war chess." In fact, he states, "The game of chess is the oldest form of war game, and modern map maneuvers have grown out of the game of chess by a long process of evolution." This statement is not supported by other historians who refuse to make such explicit statements regarding the origin of war games. Weiner (55) makes a more conservative statement:

> Although it is not possible to be definite about the origin of war games, it seems that games very early took on a formal and abstract character. By formal is meant that there were definite rules covering what players could do and what the immediate outcomes of each action would be. By abstract is meant that the rules, the playing board, the pieces, etc., were not specific representations of real life phenomena.

Interesting accounts of these early games with many detailed examples are given by Anderson (1), Thomas (54), and Young (63).

The first noteworthy improvement in these war games took place in 1798. In an effort to achieve more realism, George Venturini, at Schleswig, introduced the "New Kriegspiel." The main innovation was the introduction of maps to replace the older game boards. These maps were divided into 3,600 squares and the pieces were moved in such a way as to resemble the ordinary marches of troops, with the configuration of the ground taken into account. The expense paid to achieve this realism was a great increase in the complexity of the game. Sixty pages of rules were required to govern the movement and the fighting of the troops. This increase in complexity led to much opposition and eventually into the division of war games into "Rigid Kriegspiel" and "Free Kriegspiel," corresponding to the opposing demands for realism and simplified playable games.

In 1824 Lieutenant von Reisswitz of the Artillery of the Prussian Guard published a set of rules with the title, "Instructions for Representation of Tactical Maneuvers Under the Guise of a War Game." This form of Rigid Kriegspiel received the official approval of von Mueffling, then Chief of the General Staff, and King Wilhelm III. Modern war gaming is essentially a continuation of the game advocated by von Reisswitz. New rules have been introduced to reflect more accurately the changing nature of war, and elaborate tables, charts, and computations are necessary to incorporate such details as troop movements and effects of fire.

The free variety of war gaming was introduced in extreme form by the work of General Verdy du Vernois. In this game the elaborate tables and computations which control the play in "Rigid Kriegspiel" are replaced by an umpire who, through experience and knowledge of the outcomes of past engagements, makes decisions according to his own views. This results in a great speed-up of the game and permits the game to be played without the elaborate preparations and computing facilities necessary for the rigid form of the game.

As the fame of the Prussian system spread, both free and rigid varieties of war gaming were introduced into other countries. In 1872 Captain Baring (54) of the Royal Artillery prepared a set of rules for a similar game which was introduced in the United States at West Point. Captain W. R. Livermore (39), U. S. Corps of Engineers, designed new technical apparatus to be used in the play of the game. He also prepared extensive tables depicting results of a variety of different type engagements under varying conditions. These tables were based on data from the Civil War (1861-1865) and the Franco-Prussian War (1870-1871). Free Kriegspiel gained in popularity in the United States with the translation of the work of von Verdy by Major Swift, U. S. Army, in 1897. Similarly, both the free and the rigid varieties were introduced into almost every major country during the latter part of the nineteenth century.

War gaming, although valuable as a training and a testing device, did little to further the analysis of the conflict situation. The first mathematical analysis of the relationship between opposing forces in battle was made by F. W. Lanchester (38, 45) in 1916.

Lanchester's Equations

Lanchester's linear law is applicable to either individual duels between members of opposing forces or combat situations where firepower is randomly directed on an area known to be occupied by the opposing force.

In the first case the differential equations of combat are:

$$\dot{x}_1 = -Ak_{21} \tag{1}$$

$$\dot{x}_2 = -Ak_{12} ,$$

where

$\dot{x}_1$, $\dot{x}_2$ = the attrition rate or the rate of change of the ODD and EVEN forces, respectively, with respect to time,

k_{12},k_{21} = the coefficient of effectiveness of ODD and EVEN units, respectively, and

A = the number of duels per unit time.

In the second case the differential equations of combat are:

$$\dot{x}_1 = -k_{21}x_1x_2 \tag{2}$$

$$\dot{x}_2 = -k_{12}x_1x_2 \ ,$$

where x_1 and x_2 are the number of surviving units on ODD and EVEN sides, respectively, at time t.

It can be seen that Equations (1) is a special case of Equations (2). In an individual duel $x_1 = x_2 = 1$ and Equations (2) become:

$$\dot{x}_1 = -k_{21}$$

$$\dot{x}_2 = -k_{12} \ .$$

When the number of duels per unit time are considered, Equations (1) results.

The solution to Equations (2) is:

$$x_1 = \frac{x_{10}(k_{21}x_{20}-k_{12}x_{10})}{k_{21}x_{20}e^{(k_{21}x_{20}-k_{12}x_{10})t}-k_{12}x_{10}} \tag{2'}$$

$$x_2 = \frac{x_{20}(k_{12}x_{10}-k_{21}x_{20})}{k_{12}x_{10}e^{(k_{12}x_{10}-k_{21}x_{20})t}-k_{21}x_{20}} \ .$$

If t is eliminated from Equations (2'), the result is

$$k_{12}(x_{10}-x_1) = k_{21}(x_{20}-x_2) \ . \tag{2''}$$

This is also a solution to Equation (1).

Two sides are said to be equally matched when the ratio of their attrition is equal to the ratio of their numerical strengths, or

$$\frac{\dot{x}_1}{\dot{x}_2} = \frac{x_1}{x_2} \ . \tag{3}$$

Substituting values of $\dot{x}_1$ and $\dot{x}_2$ into Equation (2) results in

$$\frac{k_{21}}{k_{12}} = \frac{x_1}{x_2} \ .$$

The above equality establishes that, if Lanchester's linear law is a good representation of combat, two opposing forces are equally matched when the ratio of their numerical strength is equal to the reciprocal of the ratio of their effectiveness. The ratio k_{21}/k_{12} will be referred to as the effectiveness ratio.

Lanchester's square law considers the case of concentrated forces with extended firepower. Unlike the case of the individual duel, where one man is opposed to one man, or the case of area fire where fire is not directed at a specific opponent, each participant can fire at every opponent (at least in the ideal case). Thus the differential equations which characterize the attrition rates of the opposing sides depend upon the number of fighting men and their effectiveness. Lanchester's equations for this case become

$$\dot{x}_1 = -k_{21}x_2 \tag{4}$$

$$\dot{x}_2 = -k_{12}x_1 \ .$$

The solution to Equations (4) is

$$x_1 = x_{10} \cosh t\sqrt{k_{12}k_{21}} - x_{20}\sqrt{k_{21}/k_{12}} \sinh t\sqrt{k_{12}k_{21}} \tag{4'}$$

$$x_2 = x_{20} \cosh t\sqrt{k_{12}k_{21}} - x_{10}\sqrt{k_{12}/k_{21}} \sinh t\sqrt{k_{12}k_{21}} \ .$$

The solution to Equations (4), with time eliminated is:

$$k_{12}(x_{10}^2 - x_1^2) = k_{21}(x_{20}^2 - x_2^2) \ . \tag{4''}$$

The notation used in Equations (4) and in their solution is the same as for Equations (1).

Since the solution of Equations (4) is a relation between the squares of the number of combatants, this equation is often referred to as Lanchester's square law. If the values for x_1 and x_2 from Equations (4) are substituted into Equation (2), the condition for equality of fighting strength is seen to be

$$\frac{k_{21}}{k_{12}} = \frac{x_1^2}{x_2^2}$$

Thus two opposing forces are equally matched when the ratio of their effectiveness is equal to the reciprocal of the square of the ratio of their numerical strengths. Consequently, it is more profitable to increase the number of participants in an engagement than it is to increase by the same factor the effectiveness of weapons.

Recent Developments

Although Lanchester's equations are acclaimed today as one of the greatest steps forward in the mathematical analysis of combat, very little attention was given to them until the advent of World War II. The formation of various operations research groups led to the investigation of Lanchester-type equations in the analysis of combat.

These recent developments of Lanchester-type equations can be divided into two broad categories, (a) deterministic models, and (b) stochastic models. Lanchester's original equations were deterministic and assumed a homogeneous composition of forces. That is, the forces in combat are composed of units of a single type or can be expressed in terms of units of a single type, and the outcome of a conflict is completely determined by its initial conditions.

Deterministic Models--Homogeneous Case. One of the simplest extensions of Lanchester's equations is represented in the study made by Engel (21) of the amphibious assault on Iwo Jima. Engel demonstrated the validity of Lanchester-type equations in the actual combat situation where U. S. forces captured Iwo Jima. The equations used by Engel were the original Lanchester's square law equations with an added term to represent the rate at which friendly troops entered combat. These equations can be expressed as

$$\dot{x}_1 = W_1(t) - k_{21}x_2 \tag{5}$$

$$\dot{x}_2 = -k_{12}x_1 \quad .$$

The notation is the same as in Equations (1) with the added term

$W(t)$ = the production rate or the rate at which friendly troops entered combat.

The theoretical results agreed very closely with the actual fighting strength observed as shown in Figure 1. The data used in Engel's study were obtained from the Historical Division, U. S. Marine Corps.

Engel notes that difficulties in collecting the necessary detailed data have discouraged further attempts to validate Lanchester's equations.* If a sufficiently large number of specific combat situations

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*The amount of detailed data required for model validation is illustrated by the following list of data used by Engel:

(a) the total number of friendly troops put ashore each day (no friendly troops ashore prior to the beginning of the engagement);

(b) the total number of friendly casualties each day and, separately, those killed in action;

(c) the number of enemy troops ashore at the beginning of the engagement;

(d) the time when the island was declared secure (after this time, although the battle continued, it may have done so at a different rate);

(e) the time when the engagement ended (after the island was declared secure);

(f) the number of enemy troops at the end of the engagement (zero if all were destroyed).

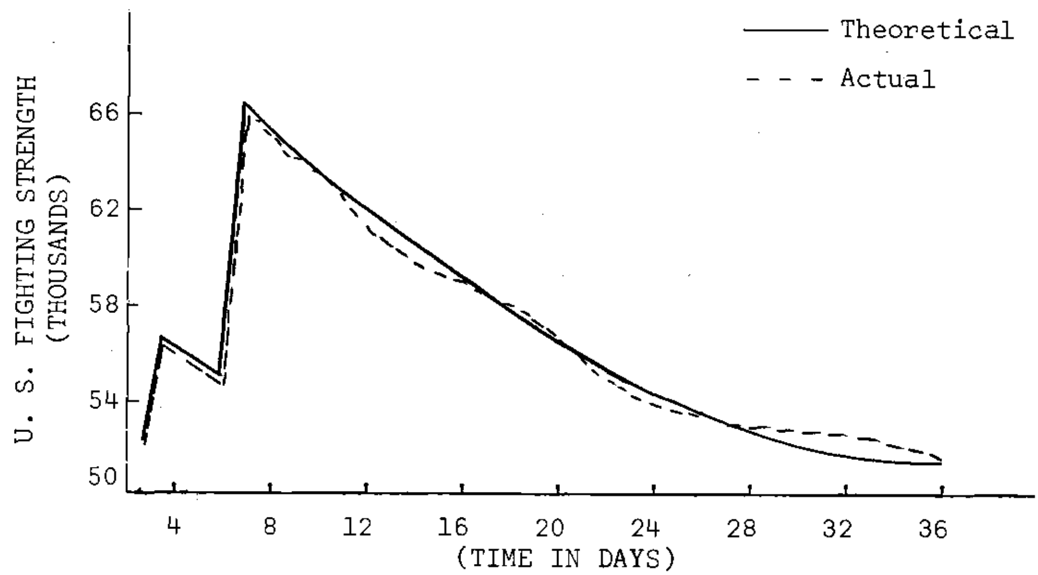

Figure 1. Theoretical versus Actual Fighting Strength During the Capture of Iwo Jima

is investigated, it may be possible to determine parameter values that pertain to certain types of combat situations as well as "operational constants" that relate these parameters to other known factors. Furthermore, such relationships will be extremely useful if it is possible to measure or estimate them prior to the inception of an engagement.

C. W. Karns (36) conducted a further study of the Iwo Jima conflict, analyzing the effect of varying W(t) on the number of casualties and the duration of the battle. The results of his study indicated that the optimum procedure in an amphibious operation of this type is to send men ashore on the first day as rapidly as ship-to-shore transport facilities and beachhead area allow, and then to re-enforce as rapidly as possible with as many troops as are necessary to keep casualties and length of battle below their acceptable maxima. This procedure was essentially followed at Iwo Jima. Karn's study further indicated that if physical facilities would have permitted the entire 73,000 men used at Iwo Jima to have been sent ashore on the first day, the battle would have been shortened by only one or two days and the casualties reduced by less than 7 per cent.

In a study of other combats, H. K. Weiss (57) observed that theoretical results based on the equations used by Engel did not agree with the empirical data but that casualty rates appeared to show a rise and fall as forces advance, withdraw, or meet prepared defenses. Weiss proposed that Lanchester's square law be modified to include the effect of the relative movement of forces. The modified equations studied were

$$\dot{x}_1 = -k_{21}x_2 \ g(r) \tag{6}$$

$$\dot{x}_2 = -k_{12}x_1 \ g(r) \ .$$

The function g(r) is monotonically decreasing function of r, the separation distance between the two opposing forces. Although the casualty rates are now functions of the separation distance between the two opposing forces, the solution of (6) with time eliminated is the same as for the original Lanchester's square law. Weiss analyzed these equations making the assumption that each commander had predetermined the maximum acceptable casualty rate and advanced or retreated accordingly. No attempt was made to verify the results of this study with data from actual combat situations. A similar analysis was previously made by

Rashevsky (49) who also included a method for introducing "morale" into the overall expressions. This paper was a theoretical treatment with no attempt at validation.

Morse (44) and others recognized the need for an additional term proportional to the number of troops engaged on a side in the conflict to represent losses of one's own troops due to misfires, accidents, and so forth. The modified equations with the additional terms added are

$$\dot{x}_1 = W_1(t) - k_{21}x_2 - k_{11}x_1 \tag{7}$$

$$\dot{x}_2 = W_2(t) - k_{12}x_1 - k_{22}x_2 \ .$$

Here the k_{11} and k_{22} coefficients represent the loss rate per man per unit time due to operational accidents. Solutions to these more general equations become increasingly difficult as shown by Morse in reference (44).

With the advent of weapons having large areas of effectiveness, modern warfare is tending toward employment of small combat groups operating independently. It became necessary to modify Lanchester's equations to represent the time rate of change of combat strength when combat takes place between such groups. Weiss (57) reformulated Lanchester's equations as

$$\dot{x}_1 = -k_{21}x_2x_1/m_1 \tag{8}$$

$$\dot{x}_2 = -k_{12}x_1x_2/m_2 \ ,$$

where m_1 and m_2 are the size of the individual groups. When $m_1 = m_2 = 1$, the solution of Equation (8) yields, after t is eliminated, the linear relation between x_1 and x_2 which is Lanchester's linear law. Weiss (57) further assumed that both sides had large weapons deliverable by air (x_{30}, x_{40}), and that each weapon inflicts casualties on a combat group in proportion to the size of the group. The equations, modified to include these terms, become

$$\dot{x}_1 = -k_{21}\, x_2\, x_1/m_1 - k_{41}\, x_{40}\, m_1 \tag{9}$$

$$\dot{x}_2 = -k_{12}\, x_1\, x_2/m_2 - k_{32}\, x_{30}\, m_2 \ .$$

Each side is assumed to choose that value of m that minimizes his instantaneous loss rate. This requirement is satisfied when

$$m_1^2 = \frac{k_{21}\, x_2\, x_1}{k_{41}\, x_{40}} \tag{10}$$

$$m_2^2 = \frac{k_{12}\, x_1\, x_2}{k_{32}\, x_{30}} \ .$$

If these values for m are substituted into Equations (9), equality of fighting strength is obtained when

$$(k_{12}\, x_1^2)\ (k_{32}\, x_{30}) = (k_{21}\, x_2^2)\ (k_{41}\, x_{40}) \ .$$

This equality is developed in detail in Appendix A. Thus the presence of weapons effective against large groups has not depressed the value of numbers of men, provided the men are employed in units of such a size as to minimize the instantaneous loss rate.

All of the studies cited have one fault in common. No theory predicting the effectiveness rates or the "effectiveness ratio," $\frac{k_{21}}{k_{12}}$, has been proposed. In 1962, R. L. Helmbold (25) conducted a detailed study of historical data with the express purpose of determining empirical relationships among various quantitative aspects of ground combat. Although Helmbold found that there apparently exists a relationship between the logarithm of the effectiveness ratio, $\ln \frac{k_{21}}{k_{12}}$, and the logarithm of the force ratio, $\ln \frac{x_{10}}{x_{20}}$, he was unable to establish bounds on this relationship.

An interesting study of a similar nature to Helmbold's was conducted late in 1962 by Willard (61). The purpose of Willard's study was to determine, by an examination of historical military data, the extent to which Lanchester's equations are an expression of a general property of battle. Willard considered both the deterministic-homogeneous case and the stochastic form of Lanchester's square law. His findings indicated that Lanchester's square law was the poorest choice of the deterministic laws. In fact, his findings indicated that casualty rates are *inversely* proportional to the square root of the opposing force. Military theorists would find this conclusion hard to believe, for it would indicate that the smaller the enemy the greater his casualty producing power and one should not strive for large armies and concentrated forces. Willard further implies that the stochastic form of the theory is a better predictor of the outcome of a battle. However, he also concludes that this model as a predictor depends upon a satisfactory estimate of the effectiveness ratio, $\frac{k_{21}}{k_{12}}$. At this time there is no satisfactory method of estimating the effectiveness ratio. Willard further concludes that "in the absence of any method of predicting E (the effectiveness ratio) reliably there is little value in a simple version of Lanchester's equations as a predictive tool where the only known quantities are the initial strengths."

Further analyses of deterministic-homogeneous Lanchester-type equations were made by Bach et al. (3) and Dolansky (16). In general, these analyses were designed to investigate

(a) the total loss of the victor as a function of the size of his fighting force;

(b) the prediction of the outcome of an engagement from initial performance data when values of the various attrition rates are not known; and

(c) analog circuits simulating the performance of the model.

Their studies all depend upon a knowledge of some factors not necessarily known prior to the inception of the battle; most generally they depend upon a knowledge of the initial attrition rates at the onset of the battle.

Deterministic Models--Heterogeneous Case. In most cases of actual combat the composition of forces is not homogeneous, i.e., forces are made up of more than one type of unit. Furthermore, it is often difficult if not impossible to assign a common measure of fighting strength to all units. The units are not commensurable. This case is usually referred to as the heterogeneous case and as yet no solution to the general heterogeneous case exists.

The differential equations for the heterogeneous case can be expressed in the same notation as used for the homogeneous case. Sides "ODD" and "EVEN" are described with subscripted notation, "ODD" having x_1, x_3, x_5, ... units of type 1, 3, 5, etc., at time t with complementary expressions for "EVEN." Thus x_1, and x_2 may represent men, x_3, x_4 aircraft, x_5, x_6 tanks, and so forth.

It should be noted that, in general, any particular type of weapon cannot attack all others but some may have the option of attacking several types of targets. Therefore, associated with each unit type is a parameter f_{ij} which represented the fraction of units of ith type which attack units of the jth type. This necessitates that the sum of all f_{ij} over j be equal to unity for all i, i.e.,

$$\sum_j f_{ij} = 1.0, \text{ for all values of } i. \tag{11}$$

Lanchester's square law for the heterogeneous case can be stated as

$$\dot{x}_{ODD} = W_1(t) - k_{11}\, x_1 - f_{21}\, k_{21}\, x_2 - k_{31}\, x_3 - f_{41}\, k_{41}\, x_4 - \cdots \tag{12}$$

$$\dot{x}_{EVEN} = W_2(t) - f_{12}\, k_{12}\, x_1 - k_{22}\, x_2 - f_{32}\, k_{32}\, x_3 - \cdots \text{ etc.}$$

Here the k_{11} and k_{22} coefficients represent operational attrition of the weapon system, non-combat illness, injuries to soldiers, and so forth.

Coefficients such as k_{31} represent unintentional attrition of own troops due to short rounds, bombing errors, and other accidental causes.

As stated before, no general solution to Equations (12) has been obtained as yet. However, many interesting limited cases have been analyzed. Noteworthy among these are the studies of Morse (44) and those of Weiss (57, 58).

Morse (44) has studied the case in which both sides have their total forces split into two types--strategic and tactical. The strategic forces are directed only against the enemy's productive capacity while the tactical forces are directed against the enemy's strategic and tactical forces. Morse further simplified the problem by assuming that the effectiveness of the strategic forces was proportional to the ratio between the strategic force and the opposing tactical force, and that furthermore the attrition coefficients for both sides were the same. This resulted in the following equations.

$$\dot{x}_1 = W_1 \left(1 - \beta \frac{x_{12}}{x_{21}} x_{12}\right) - k\,(x_2 + x_1) \tag{13}$$

$$\dot{x}_2 = W_2 \left(1 - \beta \frac{x_{22}}{x_{11}} x_{22}\right) - k\,(x_1 + x_2)\ .$$

In these equations β is the coefficient of effectiveness of the strategic units. The second subscripts on the x's pertain to the composition of forces, thus, $x_1 = x_{11} + x_{12}$ and the second subscripts 1, 2 denote tactical and strategic forces, respectively.

Assuming that the commanders of the two forces behave in a rational manner, the commander of the "ODD" side should strive to maximize the expression

$$L(x_{11},x_{21}) = \dot{x}_1 - \dot{x}_2 = W_1 - W_2 - \beta\left[W_1 \frac{(x_1-x_{11})^2}{x_{21}} - W_2 \frac{(x_2-x_{21})^2}{x_{11}}\right] \quad (14)$$

and the commander of the "EVEN" side should strive to minimize it. Since x_1 and x_2 at any instant are fixed by the previous history of the situation, the only adjustments possible are in the composition of the forces. These adjustments are made by requiring that

$$\frac{\partial L(x_{11},x_{21})}{\partial x_{11}} = 0, \quad \frac{\partial^2 L(x_{11},x_{21})}{\partial x^2_{11}} < 0 \quad (15)$$

$$\frac{\partial L(x_{11},x_{21})}{\partial x_{21}} = 0, \quad \frac{\partial^2 L(x_{11},x_{21})}{\partial x^2_{21}} > 0$$

Applying these criteria, Morse derives the optimum strategy for the two commanders. In general, the results show that the fraction of the forces which should be assigned to the tactical arm has a linear dependence on the ratio between the total forces of the two sides. It also depends on the ratio of the initial productive forces of the two sides although the dependence on this ratio is only to the one-third power. Thus, if the enemy strength increases we put more of our forces in the tactical arm; however, if our tactical forces are larger than the enemy's we can afford to put more of our strength in the strategic arm. Although

Morse's treatment of the problem is interesting, the restrictions due to his initial simplifying assumptions are such that few, if any, applications can be made of the model.

Weiss (57) conducted a similar study of a simplified heterogeneous problem. In this problem he studied the balance between air and ground forces. A flow diagram of the model is shown in Figure 2. He did not, however, consider strategic use of air power to reduce the enemy's production potential. He simplified the problem by assuming

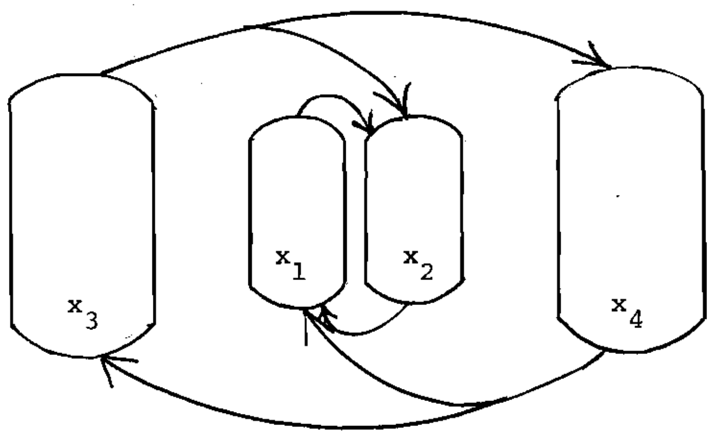

Figure 2. Flow Diagram of Weiss' Tactical Model

that ground forces can attack only ground forces, but air forces may attack either ground or air forces. He also omitted consideration of non-combat losses and force replacement and assigned equal coefficients to similar weapons on both sides.

With the above assumptions the differential equations are

$$\dot{x}_1 = -k_{21}x_2 - f_{41}k_{41}x_4 \tag{16}$$

$$\dot{x}_2 = -k_{12}x_1 - f_{32}k_{32}x_3$$

$$\dot{x}_3 = -f_{43}k_{43}x_4$$

$$\dot{x}_4 = -f_{34}k_{34}x_3 \quad .$$

Weiss solved Equations (16) for optimum tactics considering the proper balance of ground and air forces for the two opposing sides under given initial conditions. He extended the problem to include the smallest total budget to maintain force equality when the cost ratio of one unit of air to one unit of ground is known and it is possible to determine the number of units of air to replace one unit of ground without changing the outcome of the conflict.

In a later study Weiss (57) treated the problem from a differential game viewpoint following Isaacs' (30, 31, 32, 33) formulation. Weiss is careful to point out that "the model is still too great an over-simplification of real tactical problems for conclusions to be taken seriously with regard to real weapon systems."

Analog Solutions. A very simple Monte Carlo simulation of Lanchester's equations was proposed by Driggs (17). Driggs used dice with faces coded to represent casualties or attrition losses. Although the simulation was simple, sufficient flexibility was incorporated to simulate a variety of conditions. Results obtained agreed closely with analytical solutions.

Clark, et al. (13) discussed the use of electronic analogs to simulate the behavior of Lanchester's equations in the homogeneous case. Robinson (50) enlarged on the work of Clark covering a wider range of parameter values and using a slightly different analog circuit. He also formulated an analog model which combined Lanchester's square and linear laws. Bach, et al. (3) also discussed the use of analog simulation and proposed a portable unit which could be used in the field to assist commanders in arriving at strategic decisions.

Stochastic Models. Since chance enters into the actual combat situation, any deterministic model of combat such as Lanchester's equations can never exactly predict the outcome of battle. These equations only predict the approximate mean course of combat. Some excellent and comprehensive stochastic analyses of Lanchester's square law have been developed. Noteworthy among these have been the work of Snow (53) and Brown (11). Difference-differential equations (but not solutions) have been developed for the probabilities of a given number of survivors but these equations are too complex to be of much practical use. Snow (53) and Morse and Kimball (44) have shown that Lanchester's equations agree with the expected values obtained from the probability analyses as long as neither side is pushed to annihilation. Snow suggests that higher moments of the probabilistic expression may be used to establish criteria as to the confidence limits of the mean value equations.

Most authors agree that a stochastic model is preferable to the deterministic models represented by Lanchester's equations in that a stochastic model is more representative of the actual situation. However, due to the complexities involved in the probabilistic forms of the equations, no complete solutions have been obtained for the general case. There is some doubt that the additional realism achieved is worth the added complexity. Weiss (57) states:

> The probabilistic form is preferable to the simple differential equations; however it is not clear that the added difficulty of solution is consistent with the improved realism, in view of the other known variables of actual combat.

Professor Ladislav Dolansky (17) in a talk before the Tenth Anniversary Meeting of the Operations Research Society of America made a similar statement indicating that the probabilistic developments at the present time were too complicated to be of any practical use.

General Comments

Numerous Lanchester-type models of warfare have been studied and the findings of these studies recorded in the literature. A critical survey of the literature has revealed the following areas to be in need of additional research:

1. An insufficient number of validation studies have been attempted. Only one attempt has been made at a detailed comparison of Lanchester's square law against observed combat data. This study, made by Engel, showed good agreement between Lanchester's square law and combat data recorded for the battle of Iwo Jima.

2. An attempt should be made at categorizing military combat situations to determine the effect of such variables as battle size and per cent of casualties incurred.

3. There is a need for more studies of historic military data with the express purpose of determining a reliable method of estimating the effectiveness ratio, K_{21}/K_{12}. Past studies have indicated that

there is little value in a simple version of Lanchester-type equations without a reliable method of estimating the effectiveness ratio.

4. Additional research is needed, directed toward the determination of the applicability of Lanchester's square and linear laws. Possibly some exponential relation is a better model for the study of combat. There is considerable disagreement among researchers in this area.

5. The deterministic-heterogeneous case is extremely complex. No general solution exists for this case. Limited solutions which have been studied have been so restricted by initial simplifying assumptions that few, if any, applications can be made of these models.

6. The stochastic forms of Lanchester's equations are extremely complex. Due to these complexities, no complete solutions have been obtained for the general case. Most researchers doubt that the additional realism achieved by the stochastic form is worth the added complexity. More research is needed in this area, directed toward the derivation of more convenient approximate expressions for giving the number of surviving units and the probability of win.

Scope of the Study

Previous studies have stressed importance of a simple Lanchester-type expression or mathematical model of combat. General expressions or models have been too complex and simple homogeneous models have been unsuccessful due primarily to the absence of any method of reliably estimating the effectiveness ratio.

A general theory for estimating the effectiveness ratio applicable to all conceivable combat conditions would be extremely difficult if not impossible to develop. It would depend upon weapons used, targets, terrain, the type and number of troops involved, communications, etc. The development of such a general theory, therefore, is beyond the scope of this study. This study will instead, through an analysis of data on past conflicts, develop a relationship between the effectiveness ratio of the simple homogeneous model and such factors as initial strength of the two sides, and the identification of the aggressor.

In addition, an investigation will be conducted to determine whether Lanchester's square or linear law is more applicable to the study of combat. The sensitivity of the outcome to the form of the law using typical values for coefficients of effectiveness and duration of the battle will be studied.

Another area of investigation will be concerned with the categorizing of military combat situations. Categories will be formed according to total initial strengths of the combatants and the magnitude of the per cent casualties. Models will be developed representing the various combat class situations to determine the form of Lanchester law which best depicts each class of combat.

Study Objectives

The following is a chronological listing of the specific study objectives and the methods proposed for accomplishing them. The study attempts to:

1. Determine if Lanchester's square law, Lanchester's linear law or some other exponential law better depicts the flow of combat.

2. Develop empirical relationships between effectiveness ratio and initial strength.

3. Develop an advantage parameter to estimate the side having the advantage in an engagement.

4. Determine the sensitiveness of the form of Lanchester's law to the magnitude of the effectiveness coefficients.

5. Categorize military combat situations according to total force and per cent casualties and develop models to estimate effectiveness ratio and advantage for each category.

6. Estimate the validity of the models developed by determining the stability of their regression coefficients with time.

It is expected that the accomplishment of these objectives will help to establish the usefulness of Lanchester-type equations in the study of combat.

CHAPTER II

DATA

To analyze the relationships existing between factors known prior to combat and the outcome of the engagement, a comprehensive source of battle data was required. A search of the unclassified literature revealed Bodart's *Militärhistorisches Kriegs-Lexicon* (5) to be the most comprehensive source of battle data available. This work published in 1906 covers all major battles fought between the years 1618 and 1905. It is well known for its completeness and accuracy. A sample page is reproduced as Figure 3. For this study, all the sea battles were eliminated as well as all battles for which either the initial strength or the casualty data for at least one of the combatants were missing. A total usable sample of 1081 battles was obtained.

Due to the magnitude of the work anticipated, the data were transferred to punched cards.* Entries included date of the battle, names of the combatants, initial strengths, casualties of each side, page and item number in the *Lexicon,* which side was the winner, and who was the attacker. The designation of a given side as winner or loser was Bodart's and presumably reflects the judgment of the historians at the time of the compilation of the *Lexicon.* The designation of a given

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*The original set of cards was obtained from Dr. D. Willard. This deck was modified to include additional information required by this study. Copies of the final deck are obtainable on request.

562 Deutsch-Französischer Krieg 1870—1871.

1870 27./11.

SCHLACHT
bei
Amiens (5.)
(Villers-Bretonneux)

(Stadt in Frankreich, Hauptstadt des Dép. Somme, 133 km nördl. von Paris).

Sieg der **Deutschen** (30.600 Inft., 4.400 Kav., 137 Gesch. = **35.000 M.**) unter Gen. d. Inf. Fh. v. Manteuffel über die **Franzosen** (23.500 Inft., 1.500 Kav., 42 Gesch. = **25.000 M.**) unter Gen. Faidherbe.

Verluste:

	230	(1 Stb. 18 Offz.)	tot	(1 Stb. 13 Offz.)	260	
	1.070	(6 „ 50 „)	verwundet	(3 „ 33 „)	1.140	
3·7% =	1.300	(7 Stb. 68 Offz.)	**Blutige Einbusse**	(4 Stb. 46 Offz.)	1.400	= 5·6%
0·8% =	300	(— 1 „)	vermißt, gefangen	(— 20 „)	2.100	= 8·4%
4·5% =	**1.600**	(76 Offz.)	**Gesamt-Verlust**	(70 Offz.)	**3.500**	**= 14·0%**

Verl. an Trophäen: 9 Kanonen (= 21%), 2 Fahnen.

1870 15.—27./11.

BELAGERUNG
und
EINNAHME
von
La Fère

(Stadt in Frankreich, Dép. Oise, an der Oise, 22 km nordwestl. von Laon).

Die **Deutschen** (5.000 M.) zwingen die **französische** Besatzung (2.300 M., 70 Gesch.) zur Übergabe. Die Garnison wurde kriegsgefangen.

1870 28./11.

SCHLACHT
bei
Beaune-La-Rolande (4.)

(Stadt in Frankreich, Dép. Loiret, an der Rolande, 19 km südöstl. von Pithiviers).

Deutsche	**Franzosen**
GFM. Pz. Friedrich Karl v. Preußen	Gen. Crouzat

Streitkräfte:

	34.000	Infanterie	56.000	
	6.000	Kavallerie	4.000	
174 Gesch.	**40.000**	**Gesamt-Stärke**	**60.000**	Gesch. 138

Verluste:

	1.000	(1 Stb. 39 Offz.)	tot und verwundet	(11 Stb. 112 Offz.)	2.200	= 3·7%
	—	— —	vermißt, gefangen	— —	1.800	= 3·0%
2·5% =	**1.000**		**Gesamt-Verlust**		**4.000**	**= 6·7%**

1 Geschütz. **Verl. an Trophäen:**

Figure 3. Sample Page from Bodart's *Militär-historisches Kriegs-Lexicon*

side as defender or attacker is the author's. Bodart did not make this designation in his work. It is realized that in many battles, such as chance engagements, the designation of defender and attacker may be difficult, if not impossible to determine, due to attack, counter-attack, counter-counter-attack, and so forth. For the purpose of this study, the side fighting on his home soil or closest to his home soil was judged to be the defender. This rule was used consistently throughout the data.

Distribution of Data

With Time

Although the data covered the period from 1620 to 1905, the battles were not uniformly distributed in time. Figure 4 graphically depicts the distribution of the battles in time. The occurrence of periods of general military activity such as the Napoleonic Wars, the wars of Frederick the Great, and so forth, accounts for the high degree of clustering apparent in the data.

Among Countries

Countries participating in the battles covered by these data are listed in Table 1 along with their frequency of participation in number of battles over the 285-year period.

It is interesting to note that if Prussia and Germany are combined, their combined percentage is 20.4 per cent and this contribution to the entire sample is exceeded only by France and Austria. France, Austria and Prussia were without doubt the military might in Europe during most of the period covered by the sample, and it was their rivalry

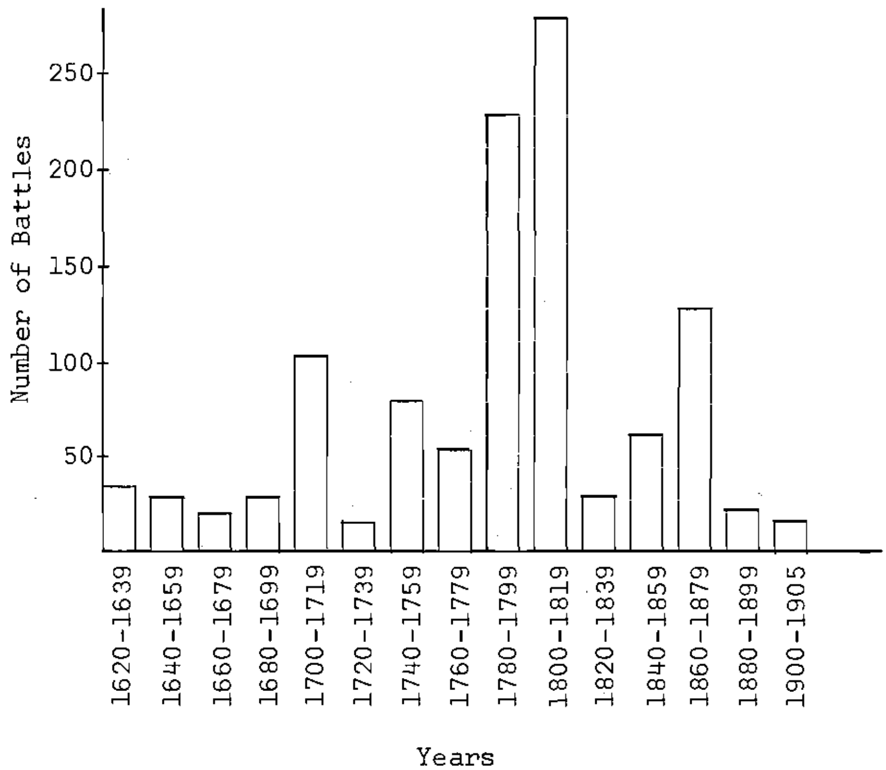


Figure 4. Distribution of Battles with Time

Table 1. Distribution of Battles by Country

Country	Battles in Which Participated	Per Cent of Data Sample
France	625	57.4
Austria	339	31.1
Russia	183	16.8
Prussia	156	14.3
Turkey	128	11.7
Great Britain	99	9.1
Holy Roman Empire	96	8.8
Germany	66	6.1
USA	58	5.3
Holland	45	4.1
Confederate States of America	43	4.0
Sweden	40	3.7
Hungary	35	3.2
Cossacks	32	3.0

which created most of the military action during this period. It is also of interest to note the contribution of the United States of America and the Confederate States during the Civil War. The United States of America, a peace loving country, contributed more than its share of bloodshed during the 150-year period.

Related Facts

Duration of Battles

Battles are defined as the operations during a period of time in which hostile forces are continually in contact with one another. Quincy Wright (62) noted that one rotation of the earth on its axis has, through most history, marked the average length of the battle. He attributed this fact primarily to the technical difficulties of fighting at night; however, due to the changes in military techniques, this observation does not hold true for the twentieth century. Over 60 per cent of all battles fought in the twentieth century have lasted longer than one day. The average duration of battles by the century is shown graphically in Figure 5.

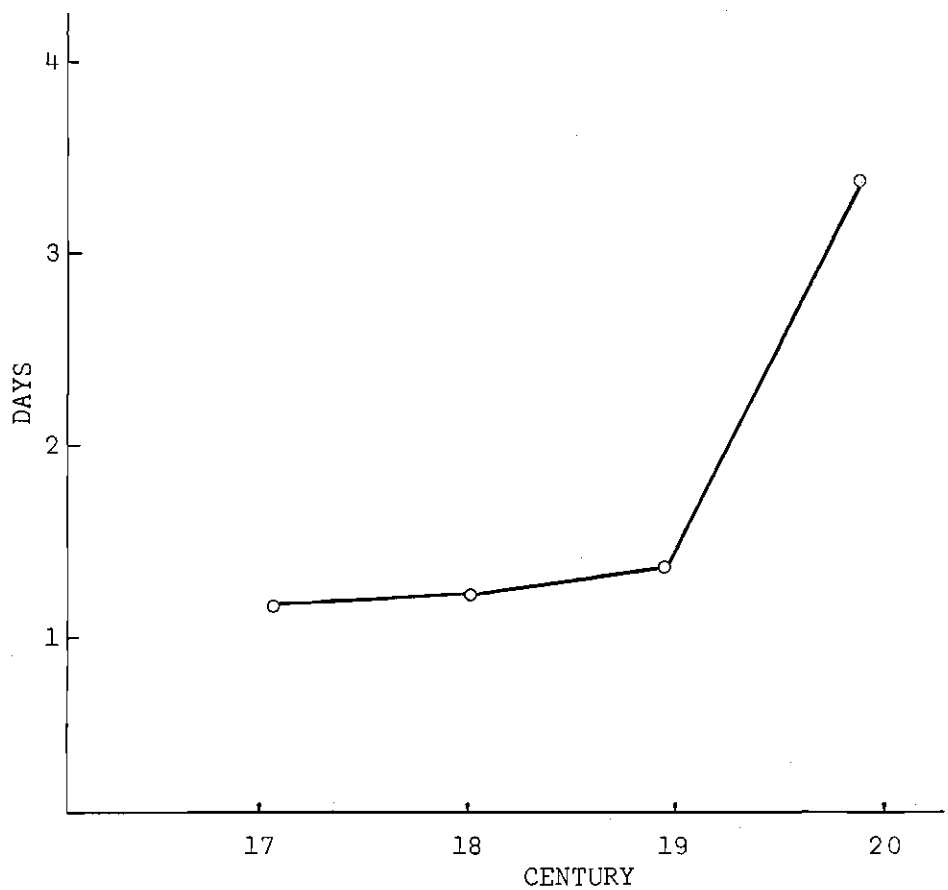


Figure 5. Average Duration of Battles by Centuries

Variation in Per Cent Casualties

There has been a noticeable trend toward a general decrease in per cent casualties with time during the period from 1630 to 1919. Tables 2-5 and Figures 6 and 7 depict this general trend. A trend toward an increase in per cent casualties would be expected due to technical improvements in the machines of destruction. No logical explanation can be given for this decrease in per cent casualties other than man's natural abhorrence to suffering. There appears to be a limit to the amount of punishment that a man is willing to endure, and with a greater ability to inflict punishment, man is apparently not willing to merely stand and fight until he is annihilated.

Table 2. Total Engaged in Battle, Total Killed, and Per Cent Casualties for France by 50-Year Periods, 1630-1919*

Time Period	Number Engaged	Number Killed or Wounded	Percentage Killed or Wounded
1630-1649	398,500	72,250	18.13
1650-1699	948,000	115,850	12.22
1700-1749	2,008,350	246,845	12.29
1750-1799	5,522,600	466,222	8.44
1800-1849	6,850,050	798,750	11.66
1850-1899	2,740,800	211,400	7.71
1900-1919	25,875,000	2,250,000	8.70
TOTAL	44,343,300	4,161,317	9.38

*Source of data for Tables 2 through 5: Quincy Wright, *A Study of War* (Chicago, 1942), pp. 658-662.

Table 3. Total Engaged in Battle, Total Killed, and Per Cent Casualties for Great Britain by 50-Year Periods, 1630-1919

Time Period	Number Engaged	Number Killed or Wounded	Percentage Killed or Wounded
1630-1649	123,000	22,000	17.89
1650-1699	446,250	68,450	15.34
1700-1749	486,850	70,780	14.54
1750-1799	883,420	54,935	6.22
1800-1849	638,700	67,685	10.60
1900-1919	14,935,706	1,115,442	7.47
TOTAL	17,689,126	1,426,422	8.06

Table 4. Total Engaged in Battle, Total Killed, and Per Cent Casualties for the United States by 50-Year Periods, 1770-1919

Time Period	Number Engaged	Number Killed or Wounded	Percentage of those Engaged, Killed or Wounded
1770-1799	72,600	8,510	11.72
1800-1849	21,400	1,810	8.46
1850-1899	4,013,700	498,242	12.41
1900-1919	6,258,000	150,284	2.40
TOTAL	10,365,700	658,846	6.36

Table 5. Total Engaged in Battle, Total Killed, and Per Cent Casualties for France, Great Britain, and the United States, 1630-1919

Time Period	Number Engaged	Number Killed or Wounded	Percentage of those Engaged, Killed or Wounded
1630-1649	521,500	94,250	18.07
1650-1699	1,394,250	184,300	13.22
1700-1749	2,495,200	317,625	12.73
1750-1799	6,478,620	509,667	7.87
1800-1849	7,510,150	868,245	11.56
1850-1899	6,929,700	736,602	10.63
1900-1919	47,398,706	3,515,916	7.47
TOTAL	72,398,126	6,226,605	8.60

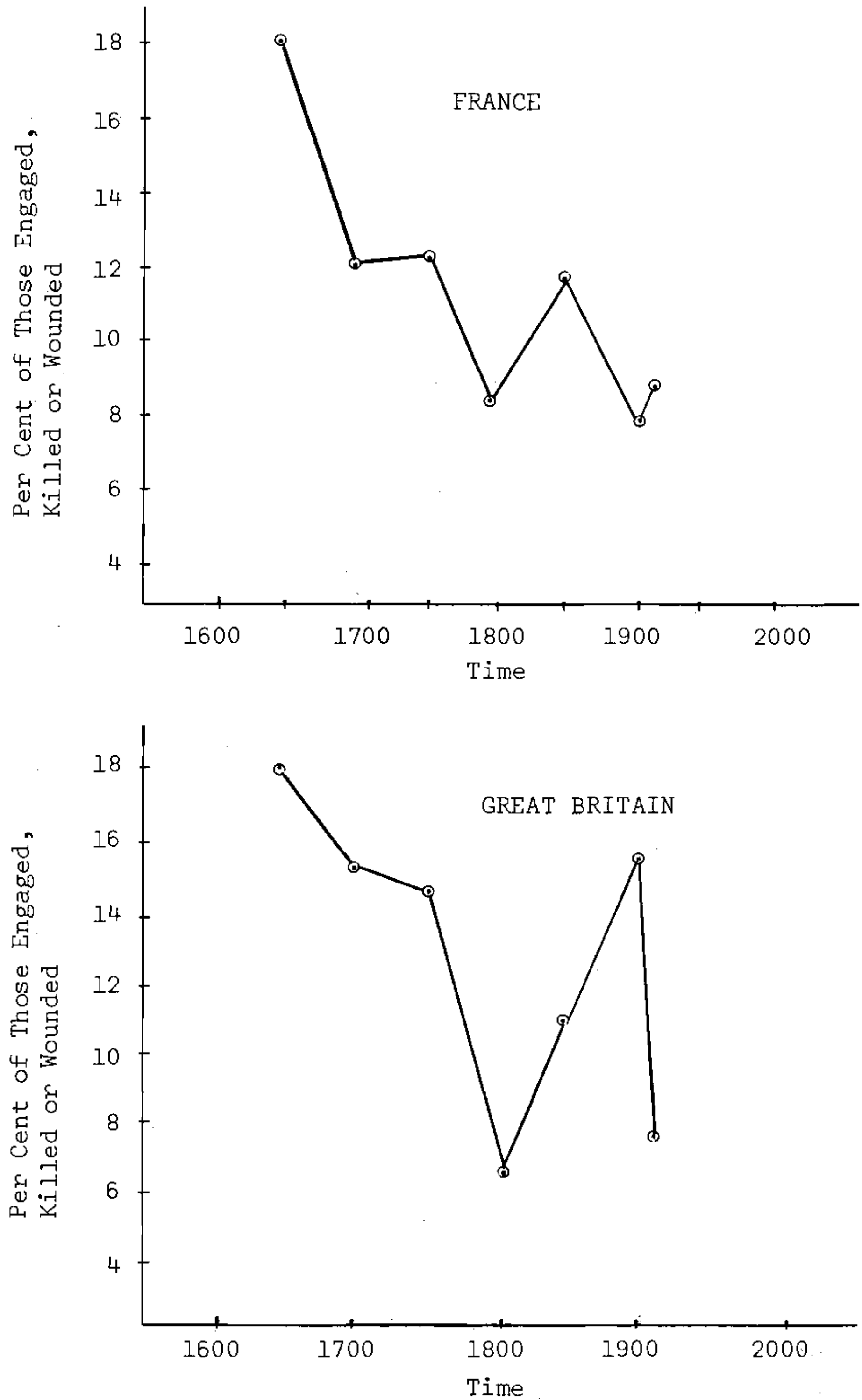
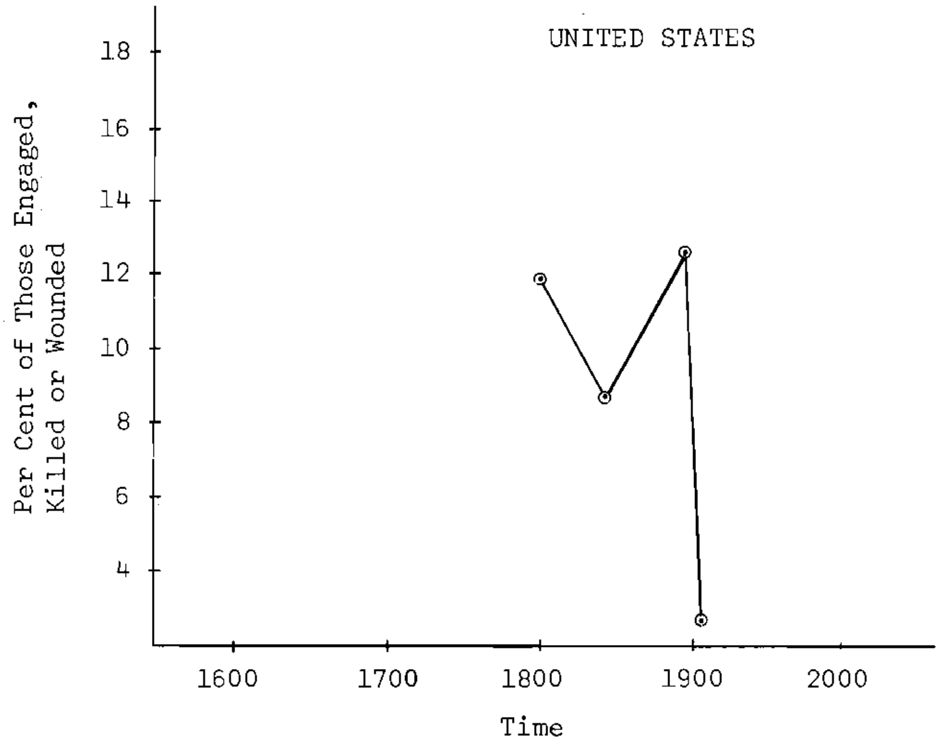


Figure 6. Per Cent of Those Engaged Who Were Killed or Wounded versus Time for France and Great Britain



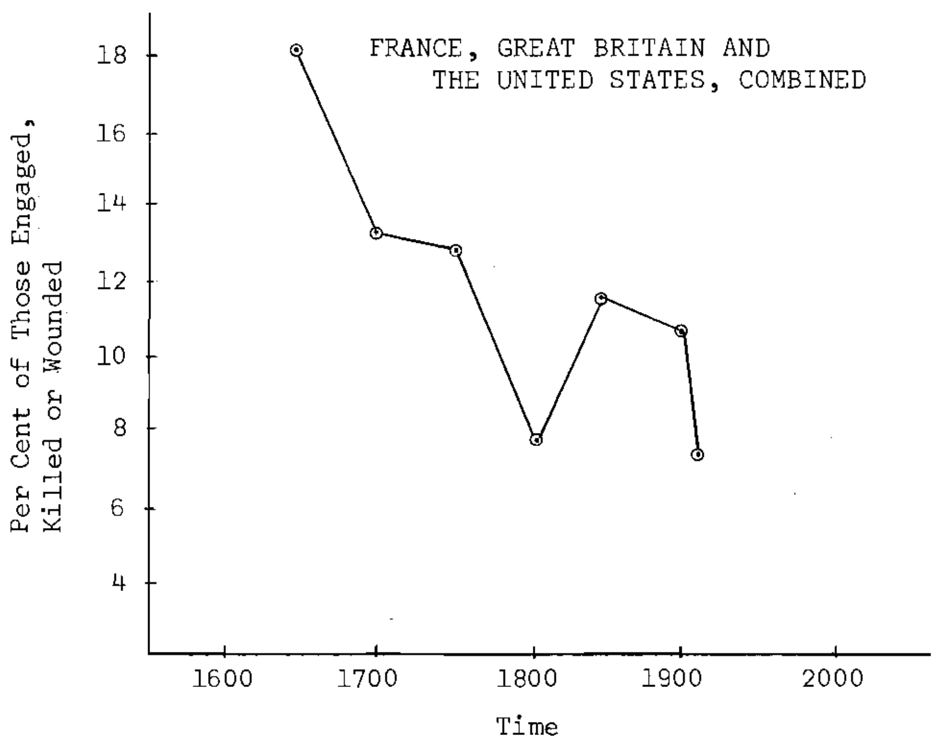


Figure 7. Per Cent of Those Engaged Who Were Killed or Wounded versus Time for the United States and All Countries Combined

CHAPTER III

PROCEDURE

It was noted in the introduction that the primary fault in the development of previous Lanchester-type models was the failure to develop either a general theory for, or a method of reliably estimating the effectiveness ratio, k_{ji}/k_{ij} $(i \neq j)$. It is an objective of this study to develop a model, or models, capable of estimating the effectiveness ratio through consideration of initial strength and identification of the aggressor and to determine the form of Lanchester laws most applicable to the data under consideration. To achieve this goal, an analysis of historical data will be conducted. A generalized form of Lanchester's equations will be used. Through regression analysis techniques, a study will be conducted to

(a) determine whether Lanchester's square law, Lanchester's linear law, or some exponential relation most nearly depicts the outcome of a military engagement, and

(b) to develop a method of estimating the effectiveness ratio.

Bodart's battle data will be categorized according to battle size and per cent casualties suffered. Separate analyses will be conducted for each category to determine the effect of battle size and per cent casualties on the outcome of the battles. A sensitivity analysis will be performed to determine the sensitivity to the forms of the law (square law or linear law) used in the analysis.

Theory

A generalized form of Lanchester's laws can be stated as

$$\dot{x}_1 = -K_{21}\ x_{10}\left(\frac{x_1}{x_{10}}\right)^{\alpha}\ x_2 \tag{17}$$

$$\dot{x}_2 = -K_{12}\ x_{20}\left(\frac{x_2}{x_{20}}\right)^{\alpha}\ x_1 \quad .$$

The value of α determines the form of the law. If $\alpha = 0$, the equations reduce to Equations (4) with $k_{21} = K_{21}\ x_{10}$ and $k_{12} = K_{12}\ x_{20}$ and thus depict Lanchester's square law. If $\alpha = 1$, the equations reduce to Equations (2) with $k_{21} = K_{21}$ and $k_{12} = K_{12}$ and thus depict Lanchester's linear law. For intermediate values of α the equations represent a condition which is neither representative of the linear law nor the square law but may be intermediate to a pure case. For example, if a battle consisted of many small engagements and if some engagements satisfied the requirements of the linear law while others satisfied the requirements of the square law, then collectively the battle should satisfy an intermediate condition and an optimum value of α between $\alpha = 0$ and $\alpha = 1$ should best represent the overall battle. For values of $\alpha > 1$, the equations depict the situation described by Peterson (48) and Weiss (59). Peterson refers to it as the "logarithmic law." In this situation a side's loss increases with the force committed. Peterson suggests that this may be due to the fact that the vulnerability of a force as a target increases directly with the force committed but that their effectiveness in delivering firepower increases at a somewhat lesser rate.

A simple flow diagram of the generalized model is shown in Figure 8.

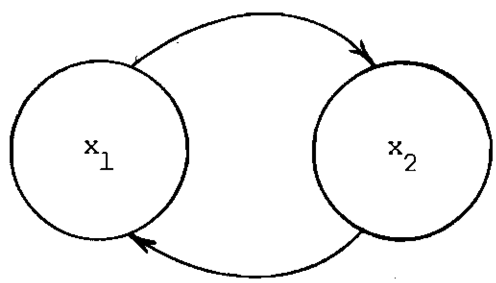


Figure 8. Flow Diagram for Simple Homogeneous Model

It should be noticed that the flow diagram for this model differs from the flow diagram for Weiss' Tactical Model shown in Figure 2. Since we are considering a homogeneity of forces, the blocks for x_3 and x_4 have been eliminated.

A solution to Equations (17) with time eliminated is

$$(x_{10}{}^{2-\alpha} - x_1{}^{2-\alpha}) = \frac{K_{21} x_{10}{}^{1-\alpha}}{K_{12} x_{20}{}^{1-\alpha}} (x_{20}{}^{2-\alpha} - x_2{}^{2-\alpha}) \ . \tag{18}$$

This equation may be rearranged to give

$$\frac{K_{21}}{K_{12}} = A(x_1, x_2) \frac{x_{10}}{x_{20}} \ , \tag{19}$$

where

$$A(x_1,x_2) = \frac{1 - (x_1/x_{10})^{2-\alpha}}{1 - (x_2/x_{20})^{2-\alpha}} \quad . \tag{20}$$

Equation (20) is not defined at t = 0, when $x_1 = x_{10}$ and $x_2 = x_{20}$.

From Equation (20) it can be seen that $A(x_1,x_2)$ is equal to the ratio of the fractional depletion of the fighting strength of the two sides. If $A(x_1,x_2)$ is less than one, the fighting strength on the "ODD" side has been depleted to a lesser extent than the fighting strength on the "EVEN" side and "ODD" has the advantage, or, more completely expressed, if

$$A(x_1,x_2) \begin{cases} < 1, \text{ ODD has the advantage.} \\ = 1, \text{ the two sides are equal.} \\ > 1, \text{ EVEN has the advantage.} \end{cases}$$

The expression $A(x_1,x_2)$, or an estimate of it, will be used throughout this study to indicate relative advantage.

Helmbold defines an expression similar to Equation (20) applicable to Lanchester's square law with x_1 and x_2 replaced with x_{1f} and x_{2f}. x_{1f} and x_{2f} are the terminal value of x_1 and x_2, respectively. Thus,

$$A(x_{1f},x_{2f}) = \frac{1 - (x_{1f}/x_{10})^{2-\alpha}}{1 - (x_{2f}/x_{20})^{2-\alpha}} \tag{21}$$

is defined as the advantage parameter of the generalized form of Lanchester's law. It is obvious that x_{1f}/x_{10} and x_{2f}/x_{20} represent the ratio of the terminal strength to the initial strength for the two sides, respectively. In the case of Lanchester's linear law, where $\alpha = 1$, Equation (21) is merely the ratio of the per cent attrition of the two opposing sides at the termination of the conflict. It is worthwhile to note that if $A(x_{1f},x_{2f})$ is greater than unity, "EVEN'S" per cent attrition is less than "ODD" and vice versa. For all values of α, the advantage parameter as represented by Equation (22) is a measure of the relative advantage of the two opposing sides. It is obvious that if the battle is fought to the annihilation of one side, the winner would be determined by the magnitude of $A(x_{1f},x_{2f})$. With $A(x_{1f},x_{2f})$ less than unity "ODD" would always be the winner; and if $A(x_{1f},x_{2f})$ were greater than unity, then "EVEN" would be the winner.

If the logarithm is taken of both sides of Equation (19), we have

$$\log (K_{21}/K_{12}) = \log A + \log (x_{10}/x_{20}). \tag{22}$$

This equation suggests that perhaps an estimate of $\log (K_{21}/K_{12})$ may be obtained through a regression analysis of historic data on the initial strength and the eventual outcome of battles. More specifically, a regression analysis will be performed to fit Bodart's data to an equation of the form

$$\log (K_{21}/K_{12}) = C_1 + C_2 \log (x_{10}/x_{20}). \tag{23}$$

The above expression may then be substituted into Equation (22) and the following equations result:

$$\log A = C_1 + (C_2 - 1) \log (x_{10}/x_{20}) \ , \tag{24}$$

or

$$A = \log^{-1}[C_1 + (C_2 - 1) \log (x_{10}/x_{20})] \ . \tag{25}$$

Furthermore, by allowing α to take on a range of values and observing the effect on the model, it should be possible to determine whether Lanchester's linear law, Lanchester's square law, or some other exponential relation is a better model for the study of combat.

Obviously, other factors influence the outcome of military engagements besides the initial strength of the combatants on each side. However, many of these factors are interdependent upon initial strength and a function relating initial strength of the combatants to the probability of success of a given side would indeed be taking many of these factors into consideration.

One of these factors that the author feels should be treated separately, however, is the determination of which side is the aggressor or attacker in the engagement. The defender in many instances has the advantage of prepared fortifications, support of the populace (usually), morale instilled by the will of all species to defend its own and as a rule a much simpler logistic problem to contend with. The attacker, on the other hand, may have the advantage of surprise on his side which can at times even outweigh vast superiority of numbers as well as the items mentioned above.

Classification of Military Engagements

The desirability of categorizing military combat situations into well-defined and easily recognizable classes has been expressed by many researchers. By classifying combat situations into specific groups with similar characteristics, it should be possible to develop simple models which more nearly depict the outcome of the engagement than would be possible if one were to develop models applicable to the general combat situation. Weiss has expressed the desirability of classifying or dividing battles by battle size while Helmbold and others have suggested a division by "bitterness" or per cent attrition. In an analysis of U. S. Civil War battles, Weiss observed that battles involving more than two divisions (30,000 men) appeared to have different characteristics than smaller battles, and Willard in an analysis of Bodart's battle data observed that 40 per cent of all battles were terminated before either side suffered 10 per cent casualties. Taking these facts into consideration the author decided to classify the data used in this study in a 3 x 3 classification resulting in nine mutually exclusive categories.

Battle size was first considered. An analysis of Bodart's battle data was performed to determine the distribution of reported battles by battle size. Battle size was defined as the total number of combatants engaged in a specific battle. A statistical analysis revealed that these data followed the exponential distribution:

$$f(x) = \lambda e^{-\lambda x}$$

with $\lambda = 0.25$ and x equal to the number of combatants (in thousands) engaged in the battle. The observed distribution of battles by size and the theoretical distribution along with a χ^2 analysis of goodness of fit is given in Table 6. Figure 9 also depicts the closeness of fit of this function to Bodart's data.

Table 6. χ^2 Goodness of Fit Analysis for $f(x) = \lambda e^{-\lambda x}$; $\lambda = 0.025$

Total Force Committed (Thousands)	BATTLES Observed Frequency	Theoretical Frequency	$\frac{(O - T)^2}{T}$
10-30	389	397	0.16
30.1-50	247	240	0.20
50.1-70	139	146	0.34
70.1-90	87	89	0.04
90.1-110	50	53	0.17
110.1-130	35	33	0.12
130.1-150	26	20	1.80
150.1-170	14	12	0.33
170.1-190	11	7	2.29
190.1-210	6	4	1.00
210.1-250	2	5	1.80

$$\chi^2 = \sum_{i=1}^{K} \frac{(O-T)^2}{T} = 8.25 \qquad 8.25 < \chi^2_{.05;10} = 18.31$$

Therefore, accept the hypothesis that

$$f(x) = 0.025e^{-0.025x}$$

does not differ from the observed distribution of battle sizes.

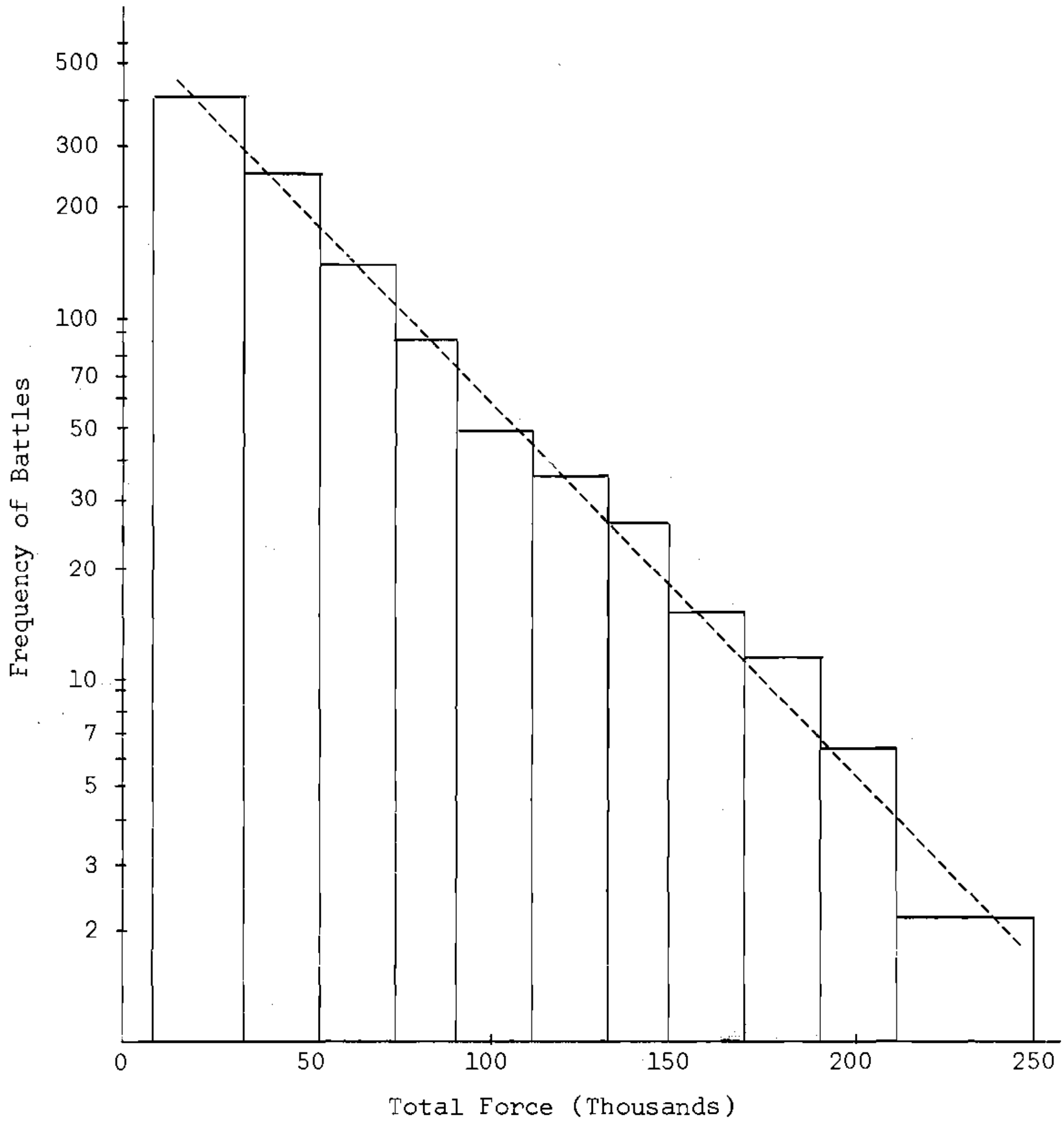


Figure 9. Frequency of Battles as a Function of Total Force Committed

Based upon this analysis of battle size and the results of Weiss' Civil War study, the battles were first divided according to those battles involving less than two divisions (30,000 men), from two to five division (30,001 to 75,000 men), and greater than five divisions. Each of these three broad categories was then divided into three subcategories by "bitterness" or per cent casualties suffered: (1) both sides suffering 10 per cent or less casualties, (2) one side having 10 per cent or less casualties with the other side having greater than 10 per cent and (3) both sides suffering greater than 10 per cent casualties.

The results of this classification are shown in Table 7, where x_{1m}, x_{2m} are the median strengths of the winner and loser, respectively, while x_{1U}, x_{2U} and x_{1L}, x_{2L} represent the upper and lower quartiles, respectively. Similarly, R_{1M}, R_{2M} represent the median per cent casualties while R_{1U}, R_{2U} and R_{1L}, R_{2L} represent the upper and lower quartiles, respectively.

Separate regression analysis will be performed on each of the nine categories as well as on the battles considered as a single class or category, i.e., all battles combined.

Three basic models will be considered for this study. All three of these models will use $A(x_1, x_2)$ as the basis for the model. The difference between the models will simply be in the manner that the initial sides "ODD" and "EVEN" are defined. In the "control model," "ODD" is defined as the side winning the engagement. In the "Initial-Force Model," "ODD" is defined as the side initially having the numerically superior force; and in the Attacker-Defender Model, "ODD" is defined as the side determined to be the aggressor.

Table 7. Summary of Battle Profiles

	$R_{i,j} \le 0.1$		$R_i \le 0.1$ $R_j > 0.1$		$R_{i,j} > 0.1$	
$x_{10} + x_{20} > 75K$	Cat 11	N = 111	Cat 12	N = 61	Cat 13	N = 62
	W	L	W	L	W	L
	x_{1U}=70K	x_{2U}=73K	x_{1U}=81K	x_{2U}=70K	x_{1U}=118K	x_{1U}=90K
	x_{1M}=52K	x_{2M}=52K	x_{1M}=65K	x_{2M}=44K	x_{1M}= 71K	x_{2M}=61K
	x_{1L}=41K	x_{2L}=41K	x_{1L}=50K	x_{2L}=30K	x_{1L}= 50K	x_{2L}=40K
	R_{1U}=0.060	R_{2U}=0.078	R_{1U}=0.100	R_{2U}=0.222	R_{1U}=0.227	R_{2U}=0.277
	R_{1M}=0.040	R_{2M}=0.050	R_{1M}=0.067	R_{2M}=0.143	R_{1M}=0.158	R_{2M}=0.200
	R_{1L}=0.020	R_{2L}=0.033	R_{1L}=0.040	R_{2L}=0.108	R_{1L}=0.121	R_{2L}=0.150
$30K < x_{10} + x_{20} \le 75K$	Cat 21	N = 174	Cat 22	N = 146	Cat 23	N = 82
	W	L	W	L	W	L
	x_{1U}=32K	x_{2U}=31K	x_{1U}=33K	x_{2U}=23K	x_{1U}=34K	x_{1U}=30K
	x_{1M}=25K	x_{2M}=22K	x_{1M}=28K	x_{2M}=17K	x_{1M}=25K	x_{2M}=21K
	x_{1L}=20K	x_{2L}=17K	x_{1L}=22K	x_{2L}=11K	x_{1L}=20K	x_{2L}=15K
	R_{1U}=0.053	R_{2U}=0.083	R_{1U}=0.088	R_{2U}=0.250	R_{1U}=0.240	R_{2U}=0.333
	R_{1M}=0.033	R_{2M}=0.054	R_{1M}=0.060	R_{2M}=0.167	R_{1M}=0.176	R_{2M}=0.235
	R_{1L}=0.020	R_{2L}=0.035	R_{1L}=0.036	R_{2L}=0.118	R_{1L}=0.135	R_{2L}=0.182

Table 7. Summary of Battle Profiles (Continued)

	$R_{i,j} \le 0.1$		$R_1 \le 0.1$, $R_j > 0.1$		$R_{i,j} > 0.1$	
	Cat 31	N = 112	Cat 32	N = 226	Cat 33	N = 107
	W	L	W	L	W	L
$x_{10} + x_{20} \le 30K$	x_{1U}=15K	x_{2U}=12K	x_{1U}=14K	x_{2U}=10K	x_{1U}=12K	x_{2U}=12K
	x_{1M}=11K	x_{2M}= 9K	x_{1M}=10K	x_{2M}= 7K	x_{1M}= 9K	x_{2M}= 8K
	x_{1L}= 6K	x_{2L}= 6K	x_{1L}= 6K	x_{2L}= 4K	x_{1L}= 5.5K	x_{2L}= 5K
	R_{1U}=0.063	R_{2U}=0.084	R_{1U}=0.083	R_{2U}=0.314	R_{1U}=0.214	R_{2U}=0.350
	R_{1M}=0.036	R_{2M}=0.070	R_{1M}=0.053	R_{2M}=0.186	R_{1M}=0.158	R_{2M}=0.250
	R_{1L}=0.020	R_{2L}=0.050	R_{1L}=0.032	R_{2L}=0.133	R_{1L}=0.125	R_{2L}=0.167

Experimental Procedure

Computer Program

Due to the magnitude of the investigation, it was necessary to perform many of the computations, as well as certain analyses with the aid of a computer. The IBM 7094 and the CDC 3600 were selected for this work due to their availability to the author. Lanchester's equations were translated into forms suitable for the type analyses performed. The regression analyses,* as well as certain of the statistical tests

*Documentation of the program which includes a flow chart, as well as a copy of the program deck used for the regression analyses, may be had upon request. The data deck is also available.

were performed on the computer. While the computer program was kept as simple as possible to conserve computer time, caution was exercised to insure that this operation would in no way infringe upon the correctness of the tests.

The Advantage Parameter

With the casualty data known at the end of the military engagement, an explicit expression can be computed for the advantage parameter $A(x_{1f},x_{2f})$ in accordance with definition (21),

$$A(x_{1f},x_{2f}) = \frac{1 - (x_{1f}/x_{10})^{2-\alpha}}{1 - (x_{2f}/x_{20})^{2-\alpha}} , \qquad (21)$$

where x_{1f} and x_{2f} represent the remaining numerical strength for "ODD" and "EVEN," respectively, at the end of the battle for which sufficient data is obtainable. The applicability of Lanchester's square law, Lanchester's linear law or some other exponential law, can then be ascertained by studying the effects on the model of varying the exponent α.

Willard (61) conducted an investigation of the casualty statistics for a large portion of the data used in this report which were reported as follows:

Table 8. Historical Incident of Casualty Ratios*

Critical Casualty Ratio R**	Fraction of Battles with R_c*** Less than R
0.05	0.13
0.075	0.25
0.100	0.41
0.125	0.50
0.200	0.73
0.250	0.82
0.333	0.90
0.500	0.98

It should be noted that 50 per cent of all battles are decided before either side loses more than 12.5 per cent of its total force. As was previously explained, the side possessing the advantage would always agree with the winning side for those battles where one side fights to annihilation.

Willard performed an analysis of these data by making use of the stochastic model proposed by Brown (11). In this analysis, Willard was able to correctly estimate the winner of the engagement in approximately 77 per cent of the battles. Both the initial strength and the casualty data were used as inputs to the model. Willard, however, was unable to

*See Willard (61), p. 18.

**$R = \frac{\text{Casualties}}{\text{Initial Strength}}$

***$R_c = \max \frac{x_{io} - x_{if}}{x_{io}}$

repeat these results when he used the advantage parameter on these same data. Both Snow (53) and Willard (61) have shown that the stochastic form used by Brown converges to the deterministic form represented by the advantage parameter. This study will use the advantage parameter to validate the theoretical work of Snow and Willard as well as to determine the value of α most applicable to the study of combat.

The Control Model

The initial force ratio for this model is defined to be the ratio of the initial numerical strength of the winning side to the initial numerical strength of the losing side, or "ODD" is defined to be the winner of the engagement. A separate bivariate regression analysis will be conducted for each specific combat category and for the composite data, making ten regression analyses in all for each value of α selected. The exponent α will be allowed to vary from $\alpha = 2$ to $\alpha = -1$.

Equations (23) and (25) will be used to estimate the effectiveness ratio and the advantage parameter for the general case as well as for the five categories. The optimum value of α will be estimated to determine the form of Lanchester's law most applicable in each case.

The Initial Force Model

In this model "ODD" is defined to be the numerically superior side, i.e., $x_{10}/x_{20} > 1$. The identification of the winning side is not needed in formulating this model. As previously stated in the theory section, the effectiveness ratio K_{21}/K_{12} can be estimated by making a logarithmic transformation of the variables and then performing a bivariate regression analysis on the transformed variable. This operation yields an analytical relationship between force ratio and effectiveness ratio and permits an estimate of the effectiveness ratio to be made with only force ratio known. This estimate can then be substituted into Equation (25) to determine the side possessing the advantage. The Initial Force Model depends upon initial force data alone to determine the side having the relative advantage in a specific engagement.

The basic equation for the Initial Force Model is

$$A = \log^{-1} [C_1 + (C_2 - 1) \log (x_{10}/x_{20})] \quad ,$$

where C_1 and C_2 are the regression coefficients of Equation (24) and are determined by the regression analyses of Bodart's data. A separate analysis is performed for each specific combat category and for the composite data, making ten regression analyses in all for each value of α selected. Each analysis results in a separate model applicable to the combat situation under study. The exponent α is allowed to vary from $\alpha = 2$ to $\alpha = -1$. The value of α is selected which tends to optimize the model.

The Attacker-Defender Model

The Attacker-Defender Model was the last model to be investigated. In this model, "ODD" is defined as the side determined to be the aggressor. It is realized that the designation of a side as an aggressor is difficult, if not impossible, in many cases, due to poor reporting of fact, attack, counter-attack, counter-counter-attack, and so forth. However, as was pointed out in Chapter II, the side fighting on its home soil, or closest to its home soil was defined to be the defender. This rule was followed consistently throughout these data.

To incorporate the datum as to which side was the aggressor into the model, it was decided to identify "ODD" to always be the aggressor or attacker in the situation. The model for the Attacker-Defender situation thus becomes

$$A = \log^{-1} [C_1 + (C_2 - 1) \log (x_{10}/x_{20})] \quad .$$

The regression coefficients C_1 and C_2 are determined by the regression analyses of Bodart's battle data. A separate analysis is performed for each specific combat category and for the composite data, making ten regression analyses in all for each value of the exponent α selected. Each analysis resulted in a separate model applicable to the combat condition under study and to the form of the law determined by the exponent α. The exponent α was allowed to vary from $\alpha = 2$ to $\alpha = -1$, and a study of the results was made to determine which value of α optimized the model.

<u>Stability of the Models with Time</u>

If the models developed in this study are to be of any use as estimators of effectiveness ratio and advantage, we must show their stability with time or how they can be expected to change with time. Since all models are based upon bivariate regressions, it becomes necessary to show the stability or trend to variation of the regression coefficients with time. This will be done by both a graphical analysis and by splitting the sample to consider discrete time periods and by analyzing the change in the coefficients over these periods of time. It is shown in the next chapter and in Appendix A that a version of the t test can be used to compare the coefficients and to determine if their variation is significant with time.

Sensitivity Analysis

To determine the sensitivity of the models to the form of the law used (square or linear law), the differential equations will be solved and surviving forces plotted with respect to time for several of the prevailing categories in the categorized data. Values of the effectiveness coefficients will be computed or estimated and allowed to vary over a range of values typical of category of combat being studied.

CHAPTER IV

ESTIMATION OF EFFECTIVENESS RATIOS AND ADVANTAGE PARAMETER

An empirical relationship will be developed between the effectiveness ratio of the generalized homogeneous model developed in the preceding chapter and the initial force ratio of the two opposing sides. It will also be shown that the advantage parameter used in the deterministic models, which are developed in this study, yields results which support the stochastic analysis conducted by Willard. Willard, in his study, used Brown's probabilistic model as a basis for predicting the winner of the engagement. A comparison of the results obtained for different values of the exponent α will be performed, and a determination will be made as to the best value of α to be used in a given model.

The Advantage Parameter

The advantage parameter, $A(x_{1f}, x_{2f})$, as defined by Equation (21) in the previous chapter,

$$A(x_{1f},x_{2f}) = \frac{1 - (x_{1f}/x_{10})^{2-\alpha}}{1 - (x_{2f}/x_{20})^{2-\alpha}} \quad , \tag{21}$$

was computed for each of the 1,081 battles contained in the sample of Bodart's battle data which was used in this study. The numerical strength of each side, both at the beginning and at the termination of the engagement, was taken directly from Bodart's data and was used in the computation of $A(x_{1f},x_{2f})$. $A(x_{1f},x_{2f})$ was then employed to estimate the side possessing the advantage in the engagement. The side possessing the advantage was then compared with the actual winner of the engagement, and the percentage of agreement was computed. In approximately 79 per cent of the battles studied, this analysis resulted in agreement between the side estimated to have the advantage and the winner of the battle.

In 1962, Willard, while a member of the Research Analysis Corporation, performed a study of the same data. In a stochastic analysis of these data, Willard utilized the probabilistic model previously developed by Brown. In this analysis, Willard correctly predicted the outcome of the engagements 76.4 per cent of the time when $\alpha = 0$ and 77.8 per cent of the time for $\alpha = 1$. These results closely parallel those obtained through the use of the advantage parameter $A(x_{1f},x_{2f})$ in this study. It is interesting to note, however, that Brown's stochastic model is slightly more sensitive to changes in the exponent α than is the author's advantage parameter.

The Control Model

A bivariate regression analysis of Bodart's battle data was used to develop an empirical relationship between the ratio of initial strengths of "ODD" and "EVEN," with "ODD defined to be the side winning the engagement, and the effectiveness ratio. Equation (23), as shown below.

$$\log (K_{21}/K_{12}) = C_1 + C_2 \log (x_{10}/x_{20}) \tag{23}$$

was used in this analysis, with values of C_1 and C_2 computed to be the least squares estimate of the slope and the intercept. The values of C_1 and C_2 were computed in such a manner that the sum of the squares of the deviations of the actual value of $\log (K_{21}/K_{12})$, as computed from Equations (18) and (19), about the fitted line

$$\log (K_{21}/K_{12}) = C_1 + C_2 \log (x_{10}/x_{20}) \tag{23}$$

is a minimum.

These computations were performed with values of α varying from $\alpha = 2$ to $\alpha = -1$. A few of the results follow. These results are listed to show how C_1 and C_2 change with changes in α.

For

$$\alpha = 2$$

$$C_1 = -0.303$$

and $$C_2 = 0.444\ ,$$

for $$\alpha = 1$$

$$C_1 = -0.283$$

and $$C_2 = 0.484,$$

for $$\alpha = 0$$

$$C_1 = 0.264$$

and $$C_2 = 0.524,$$

and for $$\alpha = -1$$

$$C_1 = -0.248$$

and $$C_2 = 0.556.$$

The small change in the regression coefficients C_1 and C_2 with changes in α reflect a general insensitivity to the form of the law. The agreement between the side possessing the estimated advantage and the winning side was identical (97%) for all values of α investigated. The computed regression coefficients for $\alpha = 1$ resulted in

$$\log (K_{21}/K_{12}) = -0.283 + 0.484 \log (x_{10}/x_{20}) \tag{26}$$

for estimating the effectiveness ratio and

$$A = \log^{-1} [-0.283 - 0.516 \log (x_{10}/x_{20})] \tag{27}$$

for estimating advantage.

It was felt that some additional information could be obtained by examining the width of the prediction interval about the regression line. Lanchester's linear law, $\alpha = 1$, was arbitrarily chosen for this investigation. Figure 10 depicts the results of the regression analyses. The dashed lines about the regression lines represent the 50 per cent prediction interval about these regression lines. This differs from confidence limits, in that confidence limits determine how well the mean value or position of the regression line is known, while the prediction interval determines the probability that an individual measurement will differ from the regression line by no more than the given amount.

In the prediction interval under consideration, the probability that some future observation of log (K_{21}/K_{12}) will be in the interval

$$C_1 + C_2 \log (\overset{o}{x}_{10}/\overset{o}{x}_{20}) \pm t_{\alpha/2;n} S_{\log(K_{21}/K_{12})|\log(x_{10}/x_{20})}$$

$$\sqrt{1 + \frac{1}{n} + \frac{[\log (\overset{o}{x}_{10}/\overset{o}{x}_{20}) - \overline{\log (x_{10}/x_{20})}]^2}{\sum_{i=1}^{n} (\log x_{10}/x_{20})_i - (\overline{\log x_{10}/x_{20}})^2}}$$

is $1 - \alpha$.* The prediction interval is more useful in the study of

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*The theory behind this prediction interval can be found in most statistics books. See Bowker and Lieberman, *Engineering Statistics*, pp. 254-255. The theory as applied in this study is developed in Appendix A.

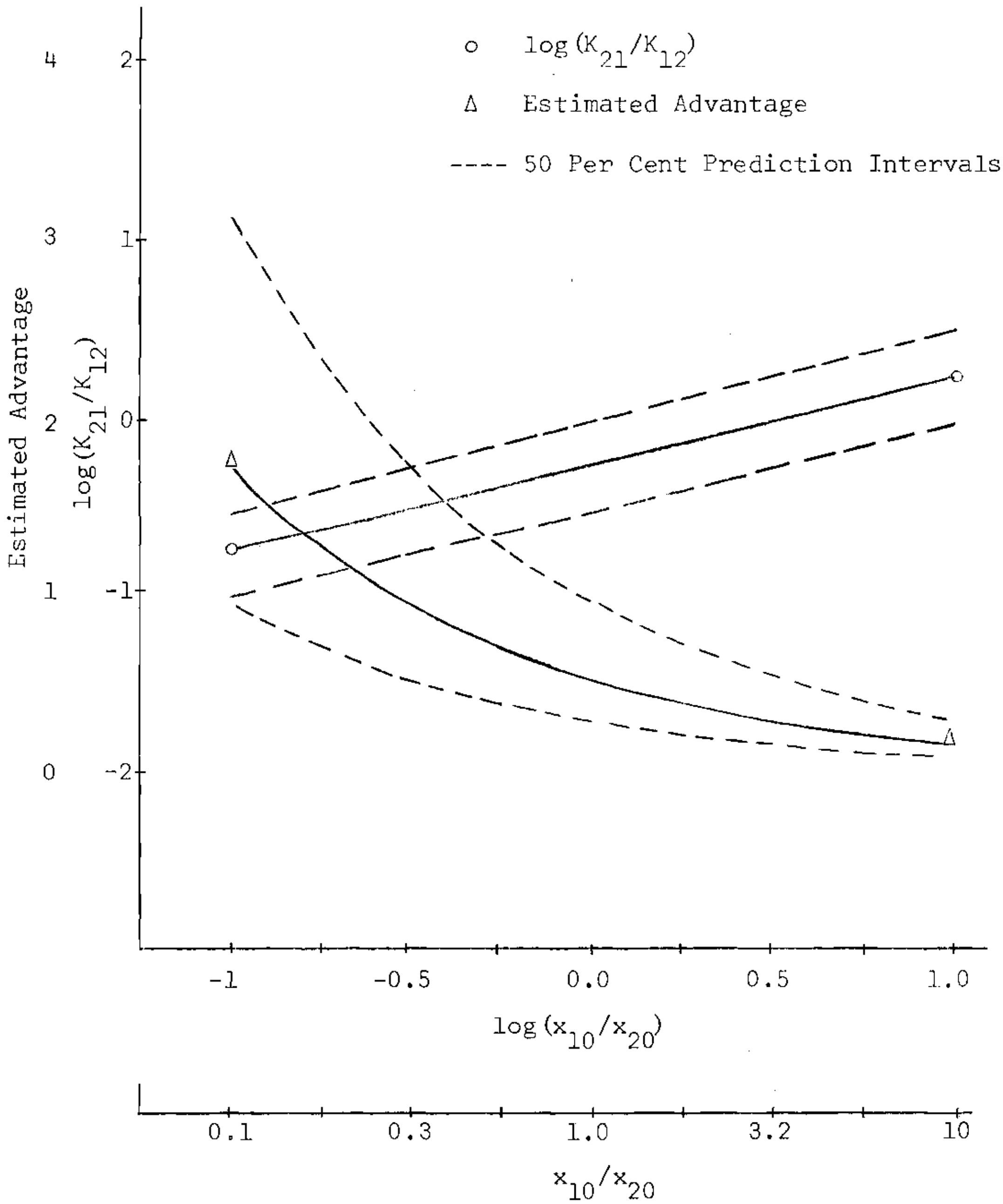

Figure 10. Results of Regression Analysis on the Control Model with α = 1.0

combat models than is the confidence interval about the regression line, since an investigator is usually more interested in how well the model will estimate individual observations.

The advantage parameter, A, is also plotted as a function of the ratios of the initial strength of the combatants. Fifty per cent prediction intervals were also computed for the advantage parameter and are shown as dashed lines about the advantage parameter line. The width of this interval is a measure of the goodness of fit of the regression line to the observed data. It should be noted that the 50 per cent prediction interval about the regression line represents an approximate 3:1 change in effectiveness ratio. For example, the average value of K_{21}/K_{20} for a unity ratio of initial strength was 0.522; however, an interval from 0.286 to 0.950 was necessary to include 50 per cent of the observed values.

As previously stated, the advantage parameter is a measure of the ratio of depletion of initial fighting strength of the two opposing forces. In the control model the ratio of initial numerical strength was formed with the winner always in the numerator. An interesting observation is that if the winner of an engagement were not known and the force ratio x_{10}/x_{20} was formed with x_{10} ("ODD") selected at random, then Equation (25) with

$$C_1 = -0.283 ,$$

$$C_2 = 0.484 ,$$

and $\alpha = 1.0$

would estimate that the winner had the advantage 97 per cent of the time for values of x_{10}/x_{20} such that

$$x_{10}/x_{20} < 0.30 \quad \text{or} \quad x_{10}/x_{20} > 3.33 \ ,$$

but for values of

$$0.30 < x_{10}/x_{20} < 3.30$$

the agreement between advantage and the winning side would be approximately 50 per cent, or no better than pure chance. This same fact was observed by Willard in his stochastic analysis of 84 battles between the Franco-Prussian War of 1870 and the year 1905. In this analysis, he found it impossible to estimate the winner of a combat on force ratios alone if force ratios were less than four to one.

Initial Force Model

In order to circumvent the problem of how to form the initial force ratio with the winning side unknown, some fast rule must be established. One such rule for forming the initial force ratios would be always to use the largest force as the numerator, forcing the ratio to be equal to or greater than unity. This model would still consider only relations between initial force ratios as input data.

In the generalized Lanchester-type model as characterized by Equation (17), the only factors necessary in addition to the effectiveness ratio and the appropriate value of the exponent α are the ratios of initial forces. It will be interesting to use the bivariate analysis method on past data to determine the effectiveness ratio. This model, defined as the Initial Force Model, will then be used to estimate the side possessing the advantage in the individual engagements contained in Bodart's battle data. Only the initial force ratio from Bodart's data will be used as input data to the model.

Bodart's battle data were used to obtain the bivariate regression coefficients C_1 and C_2 shown in Equation (23).

$$\log (K_{21}/K_{12}) = C_1 + C_2 \log (x_{10}/x_{20}) . \tag{23}$$

The ratio x_{10}/x_{20} was always formed with the superior force as the numerator, i.e., $x_{10}/x_{20} \geq 1$. The model

$$A = \log^{-1} [C_1 + (C_2 - 1) \log (x_{10}/x_{20})] , \tag{25}$$

was then used to estimate the side possessing the advantage in each engagement. A separate analysis was performed for each of the values of α between α - 2 to $\alpha = -1$ in steps of 0.25.

The results of these analyses are depicted by the following:

For

$$\alpha = 2$$

$$C_1 = -0.05$$

and $$C_2 = 0.273 ,$$

for $$\alpha = 1$$

$$C_1 = -0.05$$

and $$C_2 = 0.361$$

for $$\alpha = 0$$

$$C_1 = -0.05$$

and $$C_2 = 0.400$$

and for $$\alpha = -1$$

$$C_1 = -0.04$$

and $$C_2 = 0.440 .$$

Again the model was insensitivie to the choice of α. An analysis of the widths of the prediction interval was made for $\alpha = 1$ (Lanchester's linear law). This analysis indicated that a range of effectiveness ratios of almost 4:1 centered about the

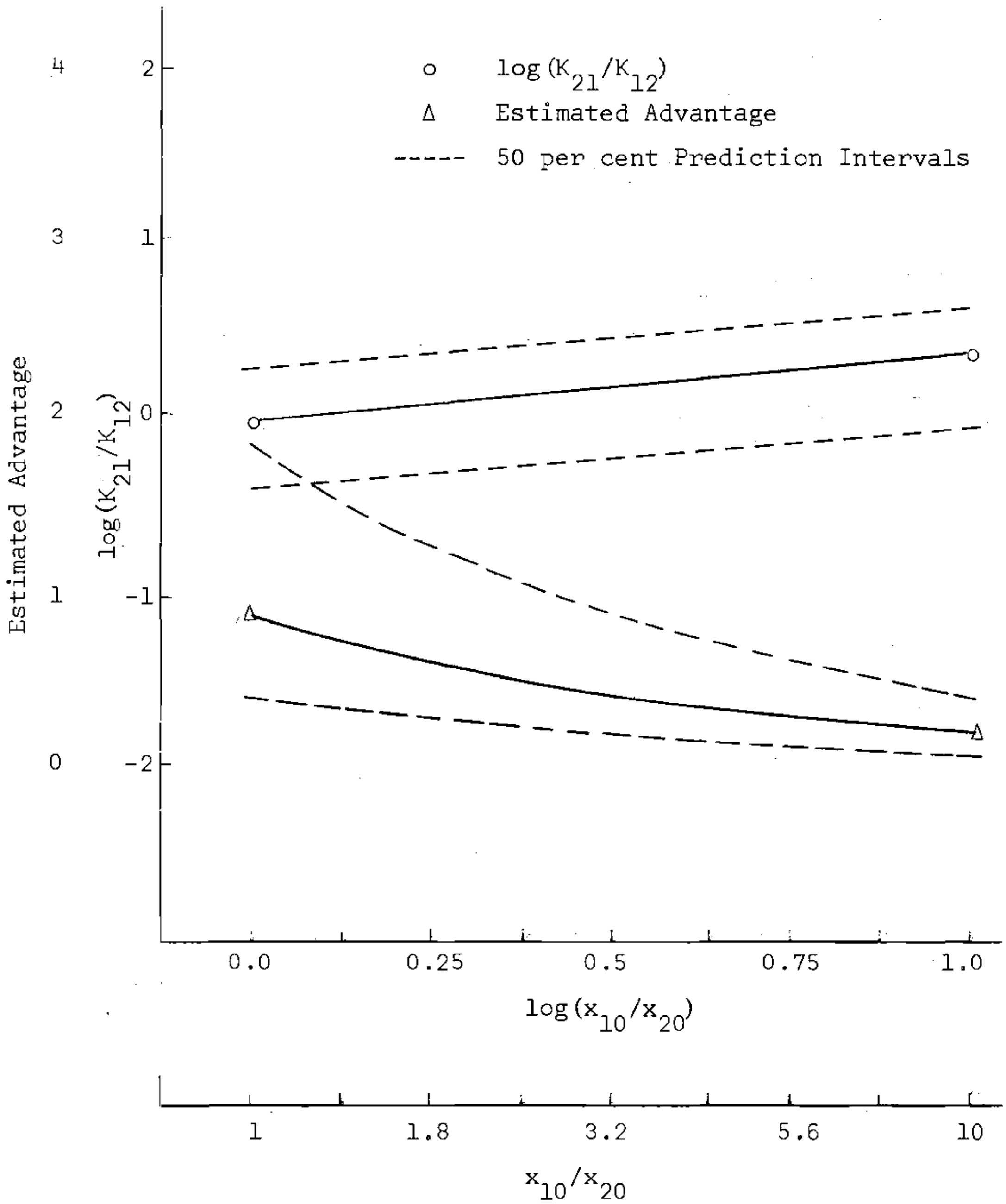

Figure 11. Results of Regression Analysis on the Initial Force Model with $\alpha = 1.0$

regression line was required to include 50 per cent of the observed data. Figure 11 illustrates the results of the regression analyses for $\alpha = 1$.

For Lanchester's linear law, the initial force model for estimating the effectiveness ratio becomes

$$\log (K_{21}/K_{12}) = -0.05 + 0.361 \log (x_{10}/x_{20}) \tag{28}$$

and for estimating the advantage

$$A = \log^{-1} [-0.05 - 0.639 \log (x_{10}/x_{20})] . \tag{29}$$

These models are illustrated in Figure 11. The side estimated to have the advantage by this model, with only initial force ratio used as an input, agreed with the winner of the engagement 66.1 per cent of the time in contrast to the 79 per cent ability for the advantage parameter which employed actual casualty data as inputs as well as initial force ratios.

Due to the small negative value of the regression coefficient C_1, it was decided to employ a t test to determine if any reason exists to believe that this coefficient was significantly different from zero. In Appendix A, a Chi Square test was used to determine if $\log (K_{21}/K_{12})$ had a normal distribution. This test gave no reason to doubt that $\log (K_{21}/K_{12})$ has a normal distribution; therefore, the t test is the proper statistic to investigate the regression coefficient C_1. The parameter t was computed as

$$t = \left| \frac{C_1}{S_{A|B} \sqrt{\frac{1}{n} + \frac{(\bar{B})^2}{\sum_{i=1}^{n} (B_i - \bar{B})^2}}} \right|$$

where

$$A = \log (K_{21}/K_{12}) ,$$

$$B = \log (x_{10}/x_{20}) .$$

This parameter has a t distribution with n-2 degrees of freedom. The t test resulted in

$$t = 1.48 < t_{.025;n} = 1.96$$

Therefore, the probability that

$$C_1 \neq 0$$

is less than 0.05 and we can simplify Equations (30) and (31) by assuming that

$$C_1 = 0 .$$

Under this assumption, the initial force model for estimating the effectiveness ratio becomes

$$\log (K_{21}/K_{12}) = 0.361 \log (x_{10}/x_{20}) , \tag{30}$$

and for estimating advantage becomes

$$A = \log^{-1} [-0.639 \log (x_{10}/x_{20})] . \tag{31}$$

The fact that the advantage parameter is equal to unity when the $\log (x_{10}/x_{20}) = 0$ graphically illustrates that this model estimates that the side with the greatest initial force will have the advantage. This estimation can be deduced from an analysis of the input to the model.

The prediction interval about the advantage parameter, $A(x_1,x_2)$, can be used to determine the probability of the smaller force having the advantage in the engagement. Figure 12 illustrates an analysis based upon Equation (31) and the prediction intervals to determine the probability of the smaller force possessing the advantage as a function of the initial force ratio. An examination of this graph reveals that a side, even though outnumbered by a factor of three to one, still has a 25 per cent chance of possessing the advantage. This fact would tend to support Willard's (61) statement that initial force alone can be a relatively weak indicator of the outcome of an engagement. It does not, however, justify his statement that a model constructed around initial force alone has little or no value.

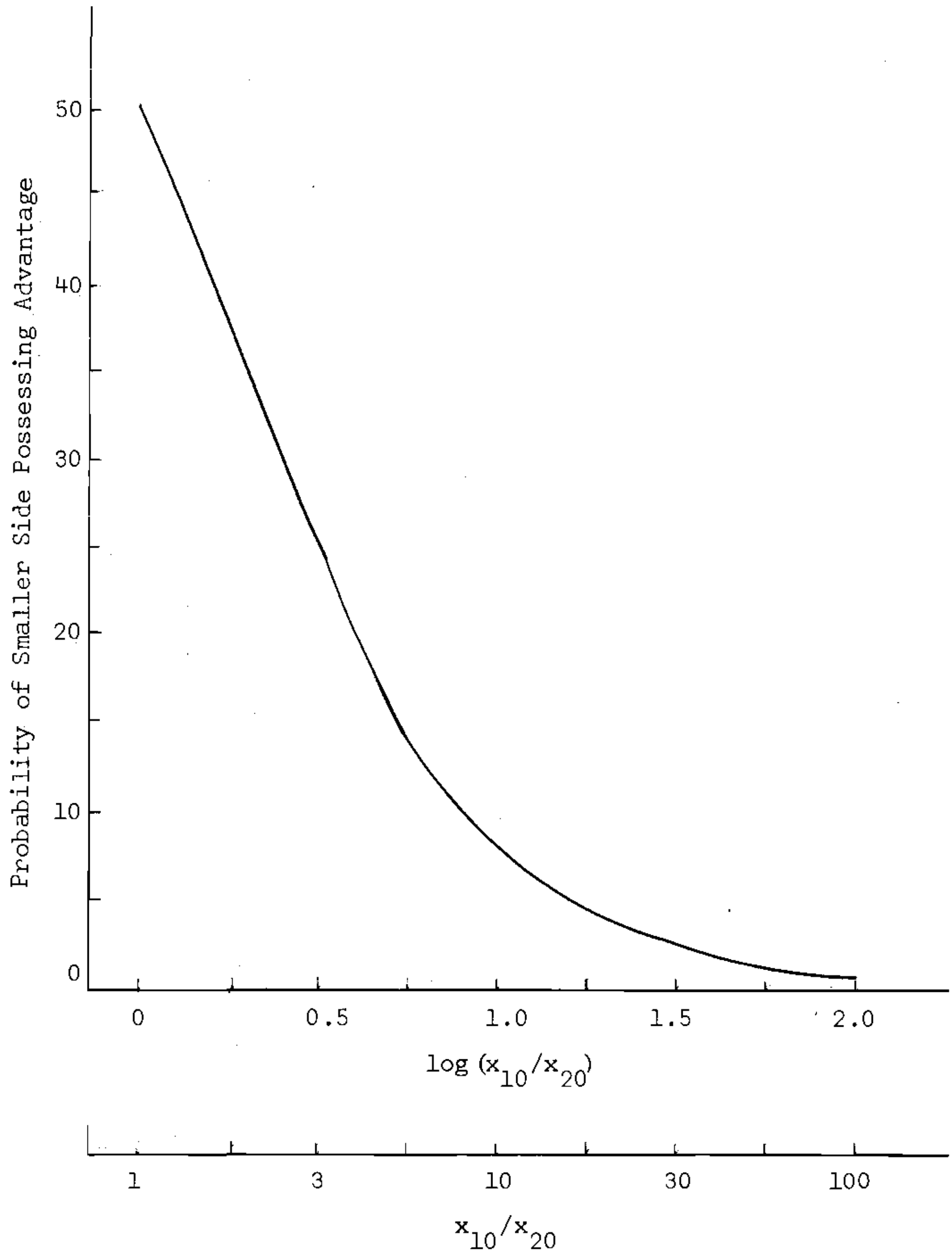

Figure 12. Probability of Smaller Side Possessing Advantage

Attacker-Defender Model

As was stated in the preceding chapter, the author contended that the defender has a relative advantage in that (a) he is fighting on his own ground, with which he is usually more familiar than the attacker, (b) he may in many instances have the advantage of prepared fortifications, (c) he usually has the support of the local citizenry and (d) he almost invariably has a much simpler logistic problem. In these times of complex war machinery, logistic problems can be quite severe and in many instances decide the flow of battle.

In order to consider the role of the attacker and the defender in the model, "ODD" was defined to be the attacker.

Bodart's battle data were again used to obtain the bivariate regression coefficients C_1 and C_2 shown in Equation (23) and the model as exemplified by Equation (25) was then used to estimate the side possessing the advantage in each engagement. As with the previous models, a separate analysis was performed for each of the values of α from A = 2 to α = -1 in steps of 0.25. The results of these analyses are depicted by the following:

For

$$\alpha = 2$$

$$C_1 = -0.008$$

and $$C_2 = 0.192 ,$$

for $\alpha = 1$

$$C_1 = -0.007$$

and $$C_2 = 0.247 \ ,$$

for $\alpha = 0$

$$C_1 = -0.006$$

and $$C_2 = 0.304 \ ,$$

and for $\alpha = -1$

$$C_1 = -0.000$$

$$C_2 = 0.349 \ .$$

As with the previous two models the results were insensitive to the choice of α.

Figure 13 illustrates the results of the regression analysis for $\alpha = 1$. An analysis of the widths of the prediction intervals indicated that again a range of effectiveness ratios of almost 4:1 was required to include 50 per cent of the observed values.

For Lanchester's linear law, the Attacker-Defender model for estimating the effectiveness ratio becomes

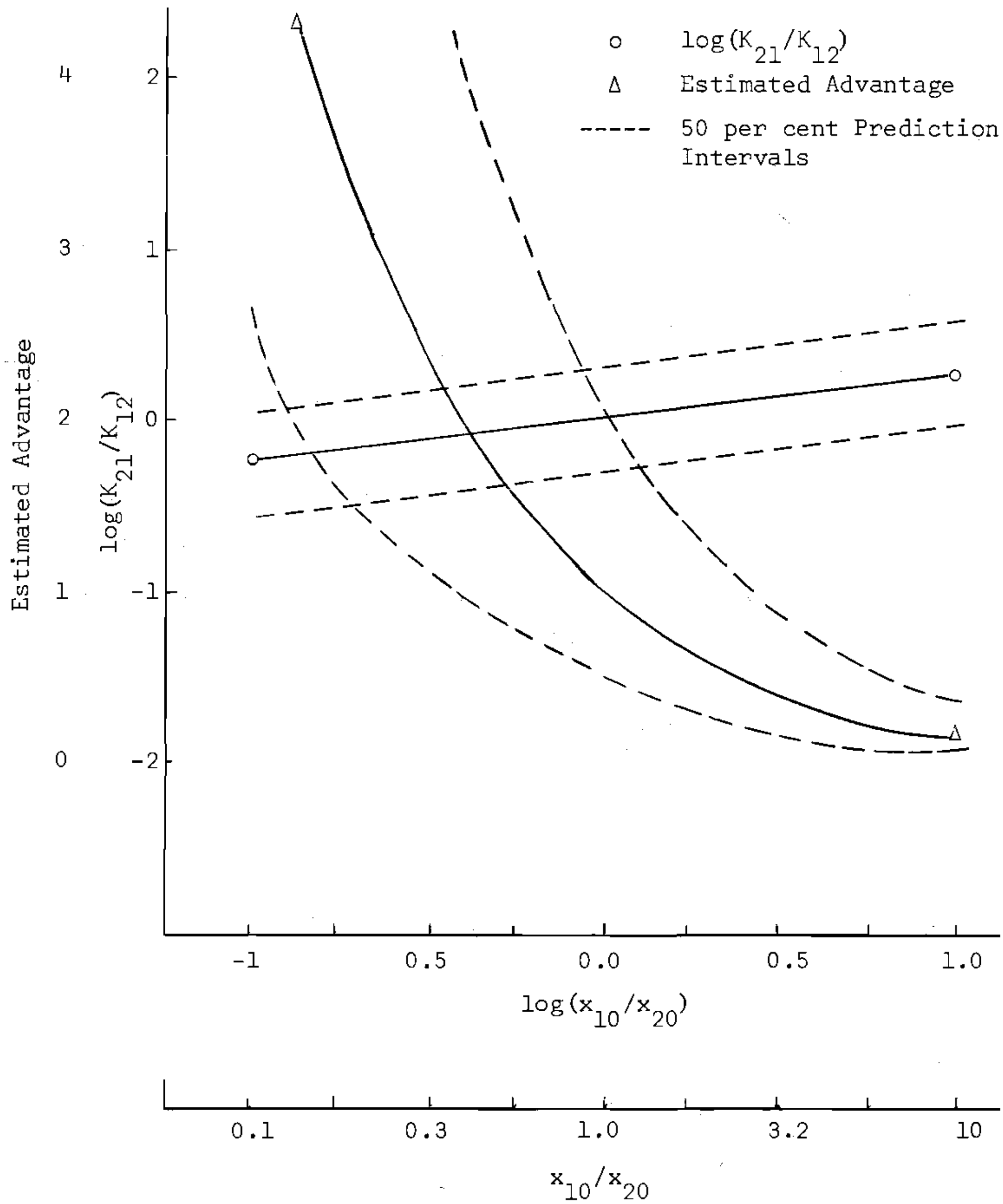

Figure 13. Results of Regression Analysis on the Attacker-Defender Model with α = 1.0

$$\log (K_{21}/K_{12}) = -0.007 + 0.247 \log (x_{10}/x_{20}) , \tag{32}$$

and for estimating the advantage becomes

$$A = \log^{-1} [-0.007 - 0.753 \log (x_{10}/x_{20})] . \tag{33}$$

Again due to the small negative value of the regression coefficient C_1, a t test was employed to determine if this coefficient was significantly different from zero. The results of the t test were

$$t = 0.42 < t_{0.025;n} = 1.96 .$$

This test indicates that there is no reason to believe that C_1 is different from zero and the attacker-defender model estimating the effectiveness ratio becomes

$$\log (K_{21}/K_{12}) = 0.247 \log (x_{10}/x_{20}) , \tag{34}$$

where "ODD" is defined to be the attacker. The attacker-defender model for estimating advantage becomes

$$A = \log^{-1} [-0.753 \log (x_{10}/x_{20})] . \tag{35}$$

Figure 13 graphically illustrates both the model estimating the effectiveness ratio and the model estimating the advantage, as exemplified in Equations (34) and (35).

The percentage agreement between the side estimated to have the advantage and the winner of the engagement was the same as that obtained by the Initial Force model. There appeared to be no advantage in identifying the aggressor.

General Comments

The advantage parameter, $A(x_{1f},x_{2f})$ using both initial and final strength data was computed for all battles studied. The side possessing the advantage was compared to the actual winner of the engagement resulting in an agreement of 79 per cent. These results are in agreement with an analysis of the same data performed by Willard in 1962. Willard, however, used a stochastic model developed by Brown.

All three of the models used were insensitive to the choice of α (form of Lanchester's law used). With initial strength only used as inputs, an agreement of 66 per cent between the side estimated to have the advantage and the winner of the engagement was obtained. The control model resulted in almost perfect agreement between the side estimated to have the advantage and the winning side. However, if the winning side were unknown prior to forming the force ratio, the model is ambiguous for force ratios

$$0.30 < x_{10}/x_{20} < 3.30 \ .$$

The Initial Force Model and the Attacker-Defender Model produced almost identical results. There appeared to be no advantage in identifying the aggressor.

CHAPTER V

ANALYSIS OF CATEGORIZED COMBAT SITUATIONS

Many researchers have objected to equations of combat which treat military engagements as an entity. In the chapter of this study dealing with test procedure, the author defined nine categories or classes of combat based upon the initial numerical strengths and the per cent casualties of the two opposing sides. This chapter will analyze these nine classes of combat to determine the form of Lanchester's law (value of α in the generalized Lanchester-type equations) which is most applicable to each of the nine classes. The bivariate regression analysis used to analyze the composite Bodart battle data and to develop the models for estimating effectiveness ratio and advantage will be used to analyze the categorized data. (The Attacker-Defender model will not be used since the results of the preceding chapter indicated no difference exists between it and the Initial Force model.) Separate analyses will be conducted for values of α from $\alpha = 2$ to $\alpha = -1$ for each of the nine classes to determine the value of α most representative of the class under study.

In addition, the estimated effectiveness ratio and advantage will be computed for each battle in a class using the value of α determined to be most representative of the class under study. The agreement between estimated advantage and the winner of the battle will be computed.

Category 11

Category 11, as defined in Chapter III, is that class of military engagements involving more than 75,000 total combatants and with the battle terminating before either side suffers casualties exceeding 10 per cent of its initial force.

As was the case with the analyses of the uncategorized data, the regression coefficients did not change appreciably with changes in α. The following regression coefficient resulted with $\alpha = 1$, Lanchester's linear law:

$$C_1 = 0.034$$

and

$$C_2 = 0.374 .$$

These coefficients give

$$\log (K_{21}/K_{12}) = 0.034 + 0.374 \log (x_{10}/x_{20}) \tag{36}$$

as the Initial Force Model for estimating the effectiveness ratio, and

$$A = \log^{-1} [0.034 - 0.626 \log (x_{10}/x_{20})] \tag{37}$$

as the Initial Force Model for estimating advantage. This model is graphically illustrated in Figure 14.

The side estimated to have the advantage agreed with the winning side only slightly in excess of 50 per cent or no better than pure

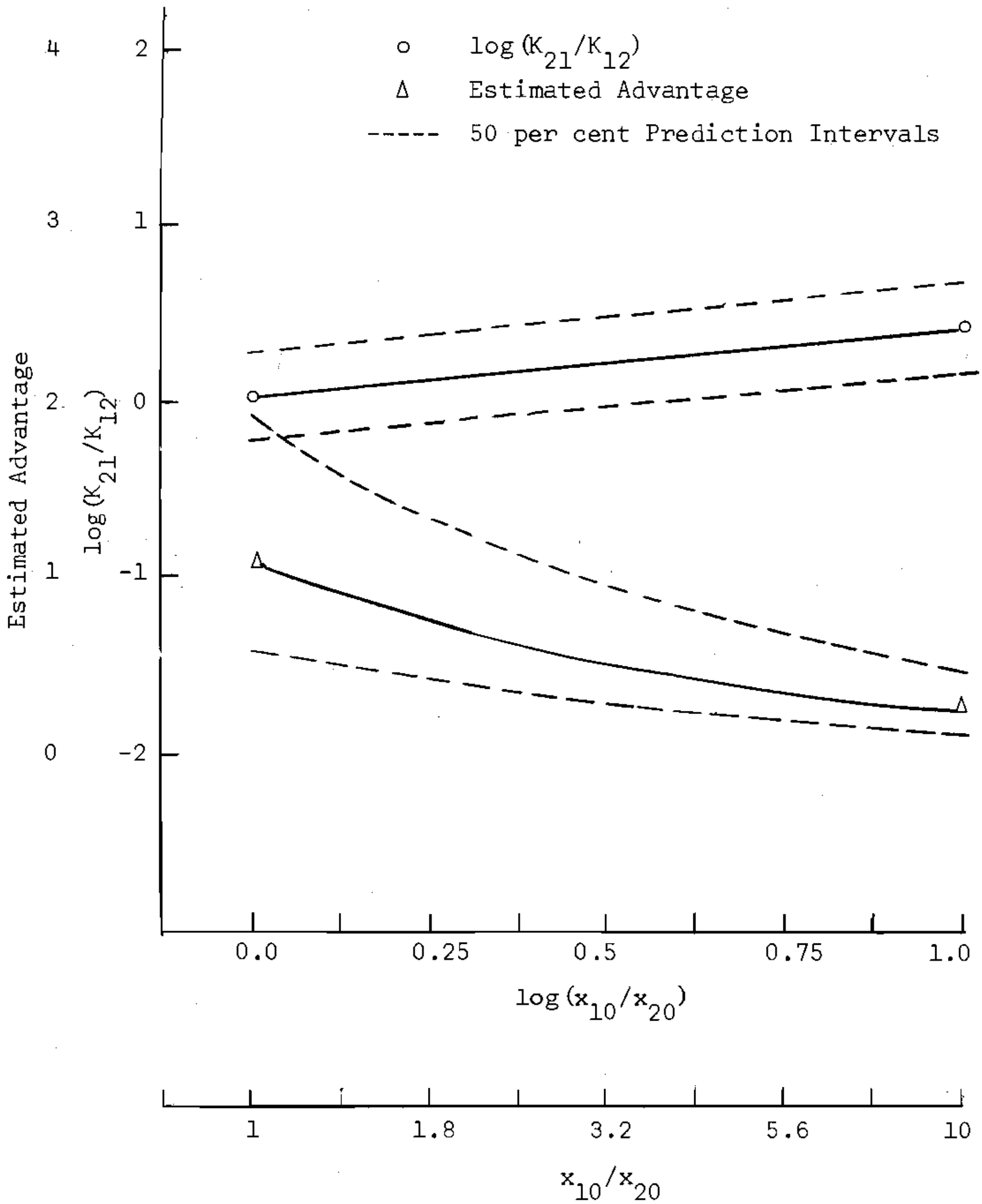

Figure 14. Results of Regression Analysis on Initial Force Model for Category 11

chance. This indicates that for this class of conflict a model based upon initial strength alone is a poor indicator or estimator of effectiveness ratio and advantage.

Category 12

Category 12 is defined as that class of military engagements involving more than 75,000 combatants and with one side suffering casualties exceeding 10 per cent, while the other side's casualties are less than or equal to 10 per cent.

The results of the regression analyses indicate no appreciable change in regression coefficients with changes in α. The following regression coefficients resulted with $\alpha = 1$, Lanchester's linear law:

$$C_1 = -0.221$$

and
$$C_2 = 0.518\,.$$

These coefficients give

$$\log\,(K_{21}/K_{12}) = -0.221 + 0.518 \log\,(x_{10}/x_{20}) \tag{38}$$

as the Initial Force Model for estimating the effectiveness ratio, and

$$A = \log^{-1}\,[-0.221 - 0.482 \log\,(x_{10}/x_{20})] \tag{39}$$

as the Initial Force Model for estimating advantage. This model is graphically illustrated in Figure 15.

The side estimated to have the advantage agreed with the winning side 70 per cent of the time indicating that for this class of conflicts the model exemplified by Equations (39) is a relatively good estimator of advantage for this category of combat.

The negative value for C_1 indicates that in general the side having the larger initial force also has the highest unit effectiveness for force ratios of less than 2.7:1. For larger force ratios the smaller force would have the higher unit effectiveness ratio. Weiss in a study of U. S. Civil War battles suggests that for large conflicts the overall force effectiveness does not increase as fast as the increase in force. These results appear to be in conflict with Weiss for ratios less than 2.7:1.

<u>Category 13</u>

Category 13 is defined as that class of military engagements involving more than 75,000 combatants and with both sides suffering casualties exceeding 10 per cent of their initial force.

Again the results of the regression analyses indicate no appreciable change in regression coefficients with changes in α. The following regression coefficients resulted with $\alpha = 1$, Lanchester's linear law:

$$C_1 = -0.051$$

and $$C_2 = 0.725 .$$

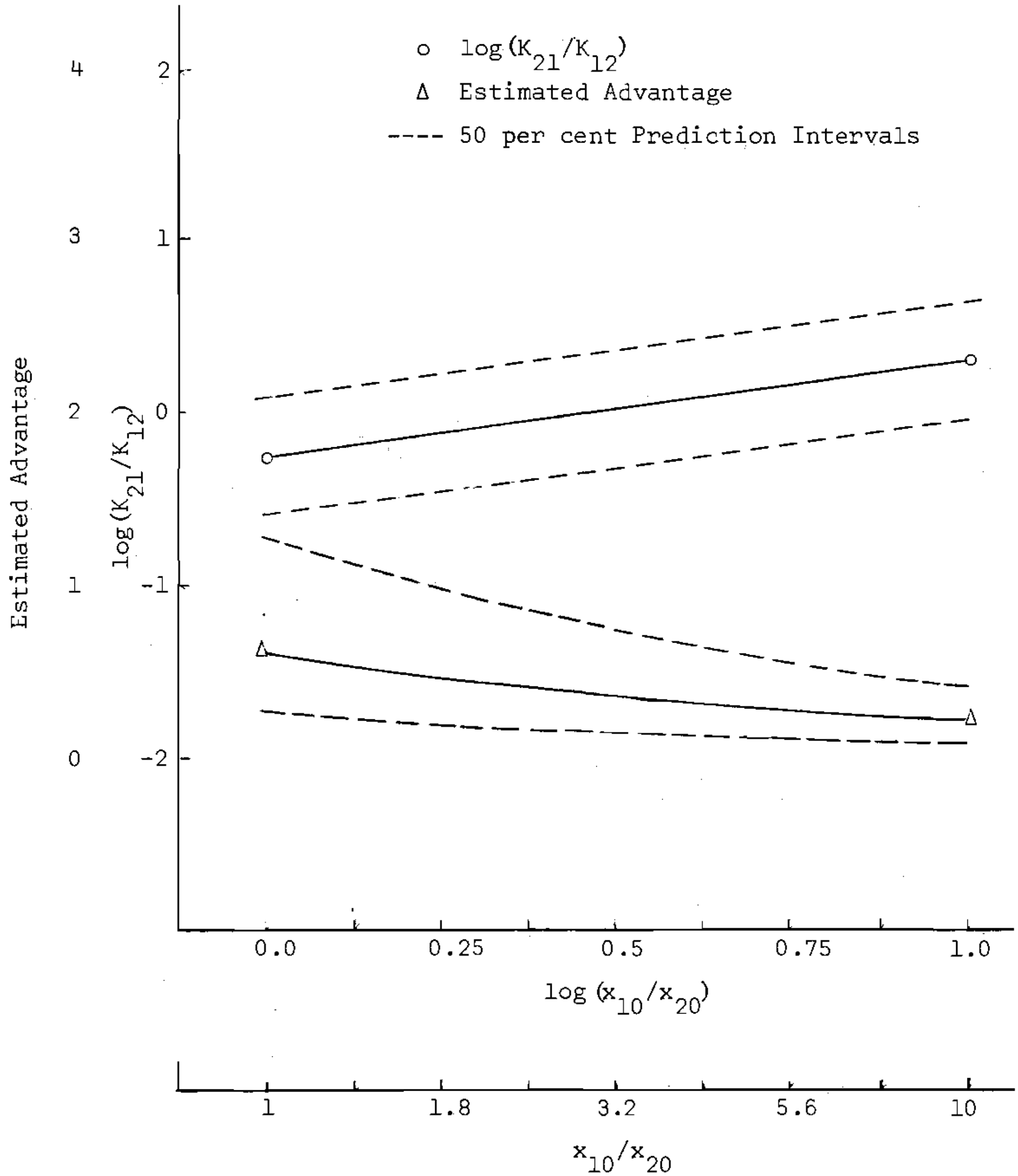

Figure 15. Results of Regression Analysis on Initial Force Model for Category 12

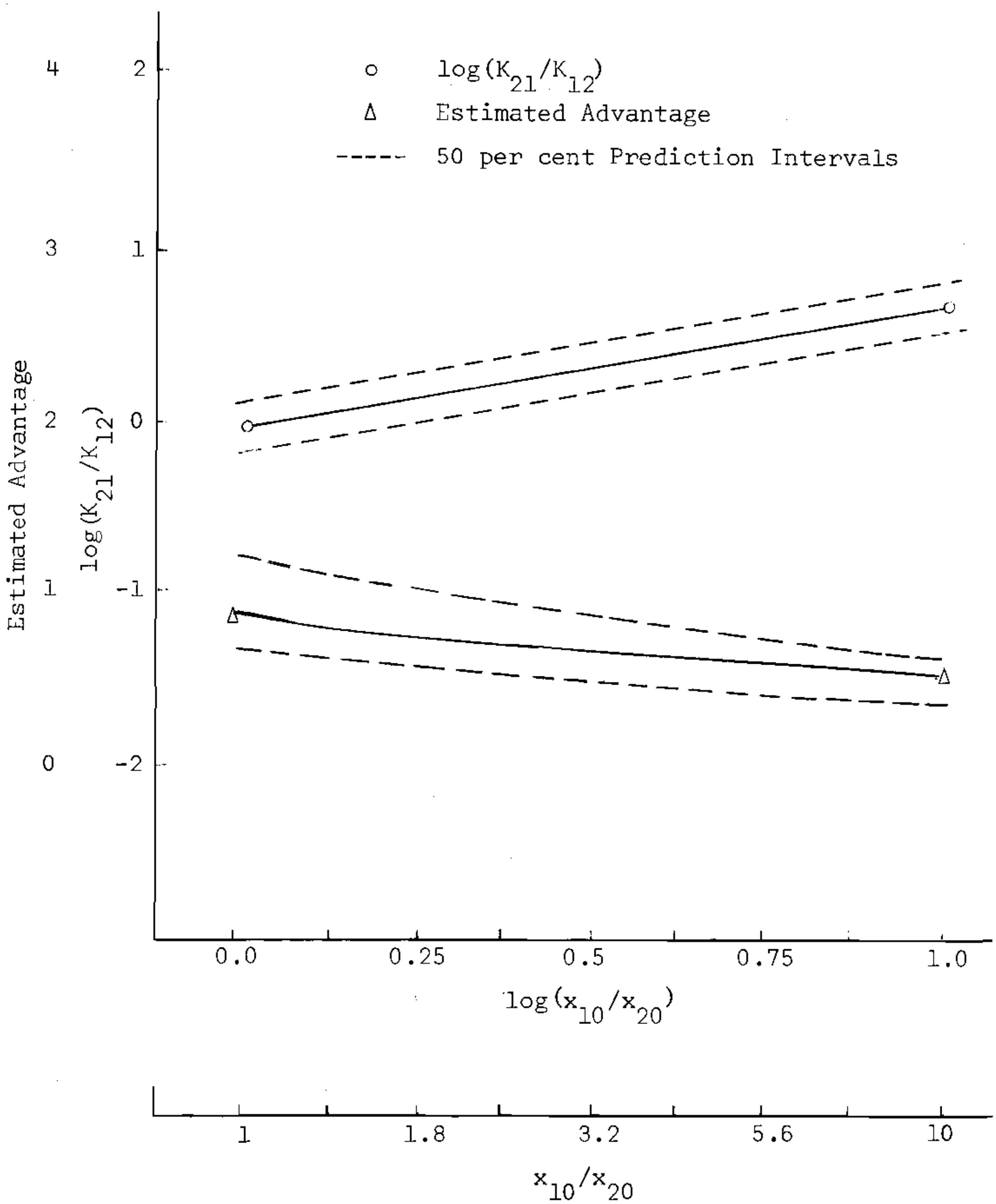

Figure 16. Results of Regression Analysis on Initial Force Model for Category 13

These coefficients give

$$\log (K_{21}/K_{12}) = -0.051 + 8.725 \log (x_{10}/x_{20}) \qquad (40)$$

as the Initial Force Model for estimating the effectiveness ratio, and

$$A = \log^{-1} [-0.051 - 0.275 \log (x_{10}/x_{20})] \qquad (41)$$

as the Initial Force Model for estimating advantage. This model is graphically illustrated in Figure 16.

The side estimated to have the advantage agreed with the winning side in 69 per cent of the 62 battles investigated.

Category 21

Category 21 is defined as that class of military engagement involving from two to five divisions (30,000 to 75,000 combatants) and with the battle terminating before either side suffers casualties exceeding 10 per cent of its initial force.

The results of the regression analyses were insensitive to the choice of α. The following regression coefficients resulted with $\alpha = 1$, Lanchester's linear law:

$$C_1 = -0.031$$

and $$C_2 = 0.448 \quad .$$

These coefficients give

$$\log\,(K_{21}/K_{12}) = -0.031 + 0.448 \log\,(x_{10}/x_{20}) \tag{42}$$

as the Initial Force Model for estimating the effectiveness ratio, and

$$A = \log^{-1}\,[-0.031 + 0.552 \log\,(x_{10}/x_{20})] \tag{43}$$

as the Initial Force Model for estimating advantage. This model is graphically illustrated in Figure 17.

The side estimated to have the advantage agreed with the winning side in 56 per cent of the 174 battles investigated or not much better than pure chance.

Category 22

Category 22 is defined as that class of military engagements involving between two and five divisions (30,000 to 75,000 men) and with one side suffering casualties exceeding 10 per cent, while the other side's casualties are less than or equal to 10 per cent.

The results of the regression analysis were insensitive to the choice of α. The following regression coefficients resulted with $\alpha = 1$, Lanchester's linear law:

$$C_1 = -0.121$$

and $$C_2 = 0.076\,.$$

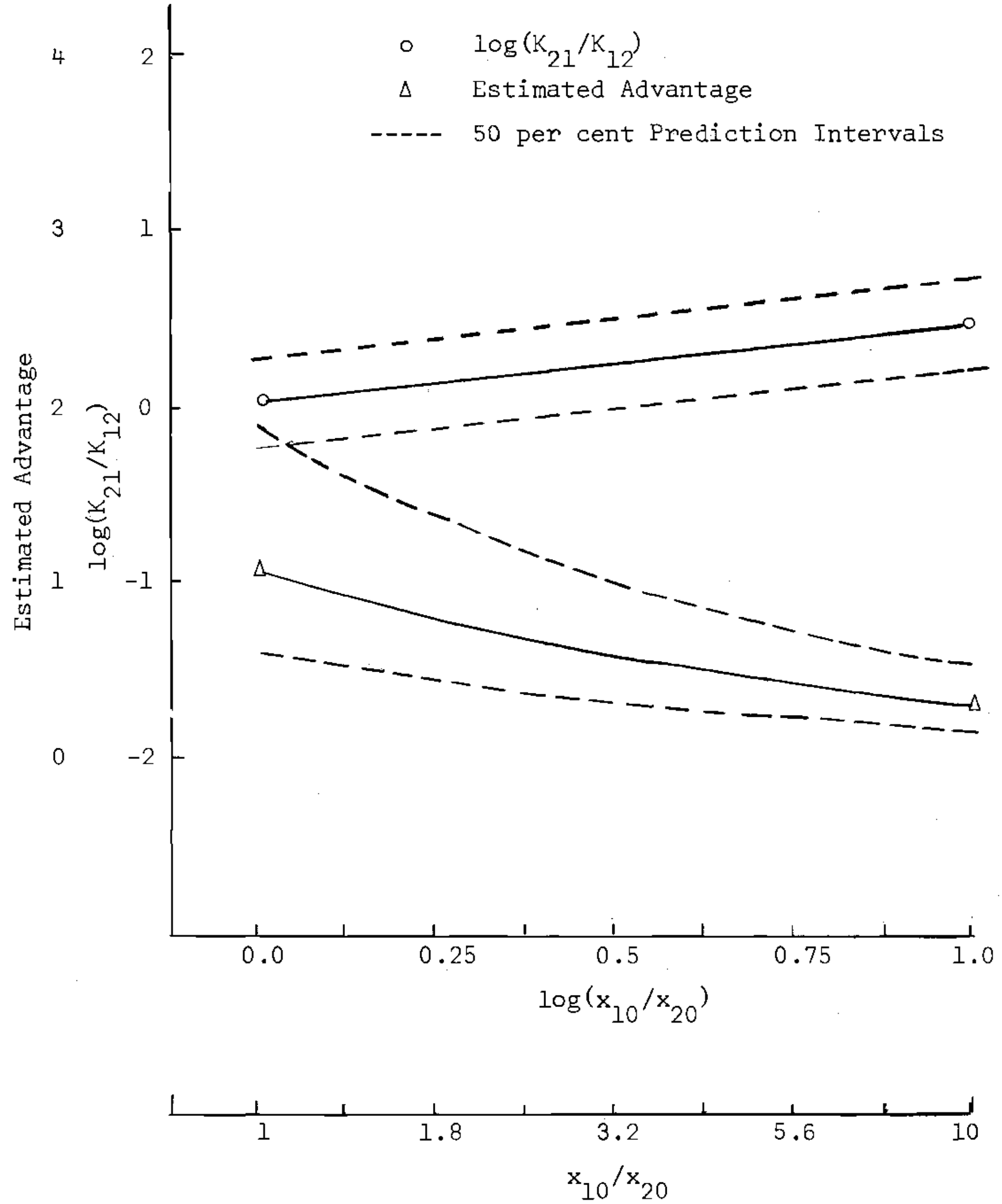

Figure 17. Results of Regression Analysis on Initial Force Model for Category 21

These coefficients give

$$\log (K_{21}/K_{12}) = -0.121 + 0.076 \log (x_{10}/x_{20}) \tag{44}$$

as the Initial Force Model for estimating effectiveness ratio, and

$$A = \log^{-1} [-0.121 - 0.924 \log (x_{10}/x_{20})] \tag{45}$$

as the Initial Force Model for estimating advantage. This model is illustrated in Figure 18.

The side estimated to have the advantage agreed with the winning side in 78 per cent of the engagements investigated. Thus, this model is considered to be a good estimator of advantage.

Again the negative value of C_1 coupled with the small value of C_2 indicates that, in general, the side having the largest initial force also possesses the highest unit effectiveness. This again appears to be in conflict with Weiss.

Category 23

This category is defined as that class of military engagements involving between two and five divisions (30,000 to 75,000 men) and with both sides suffering casualties exceeding 10 per cent of their initial force.

The results of the regression analyses were insensitive to the choice of α. With $\alpha = 1$, Lanchester's linear law, the following regression coefficients were computed:

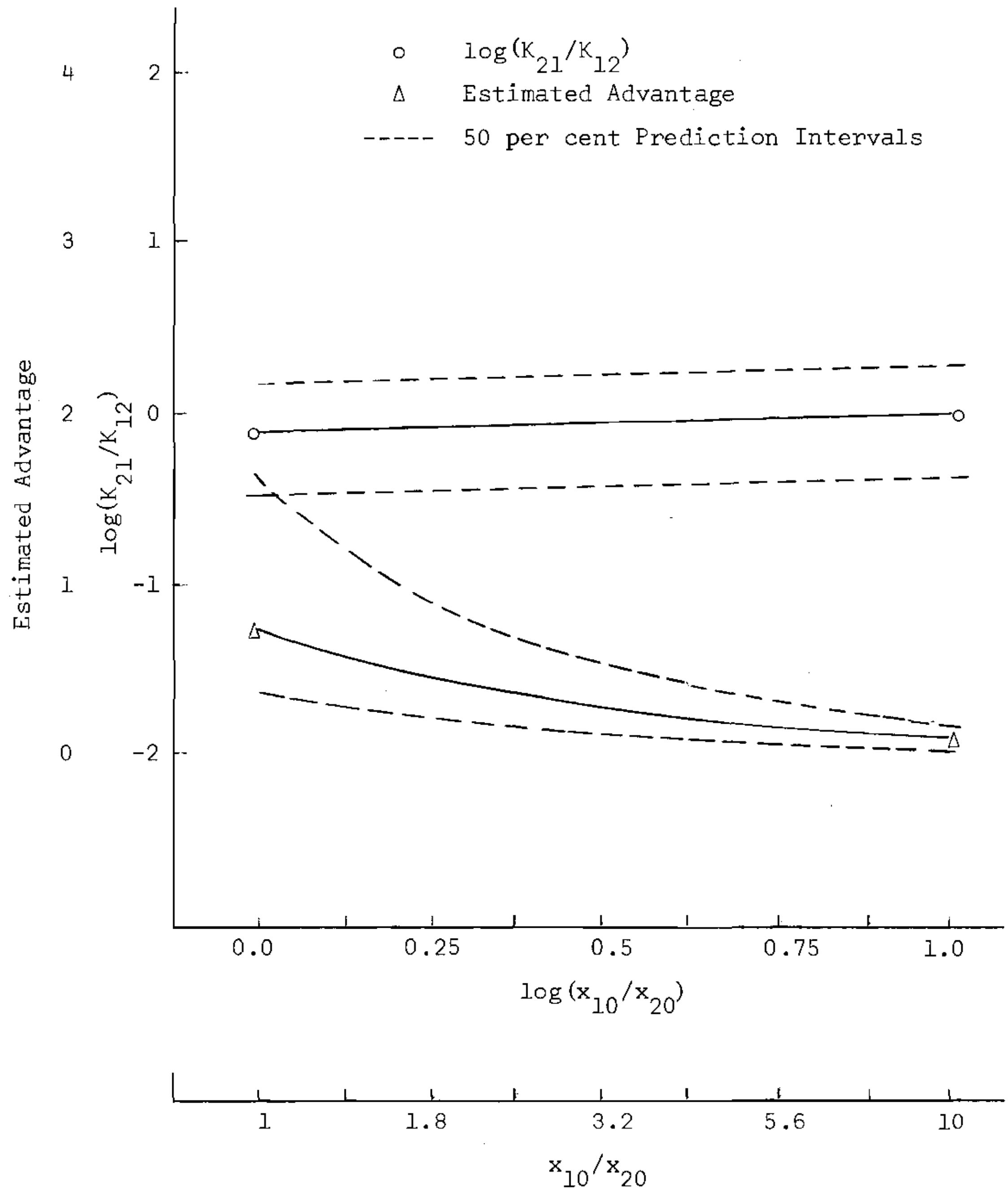

Figure 18. Results of Regression Analysis on Initial Force Model for Category 22

$$C_1 = 0.037$$

and $$C_2 = 0.594 .$$

These coefficients give

$$\log (K_{21}/K_{12}) = 0.037 + 0.594 \log (x_{10}/x_{20}) \tag{46}$$

as the Initial Force Model for estimating the effectiveness ratio, and

$$A = \log^{-1} [0.037 - 0.406 \log (x_{10}/x_{20})] \tag{47}$$

as the Initial Force Model for estimating advantage. This model is graphically illustrated in Figure 19.

The side estimated to have the advantage agreed with the winning side in 60 per cent of the battles studied.

Category 31

The category is defined as that class of military engagements involving no more than two divisions (30,000 men) and with the engagement terminating before either side suffers casualties exceeding 10 per cent of their initial force.

The regression analyses were insensitive to changes in α and Lanchester's linear law was arbitrarily chosen to be representative of this class of combat. The regression coefficients computed for $\alpha = 1$ were

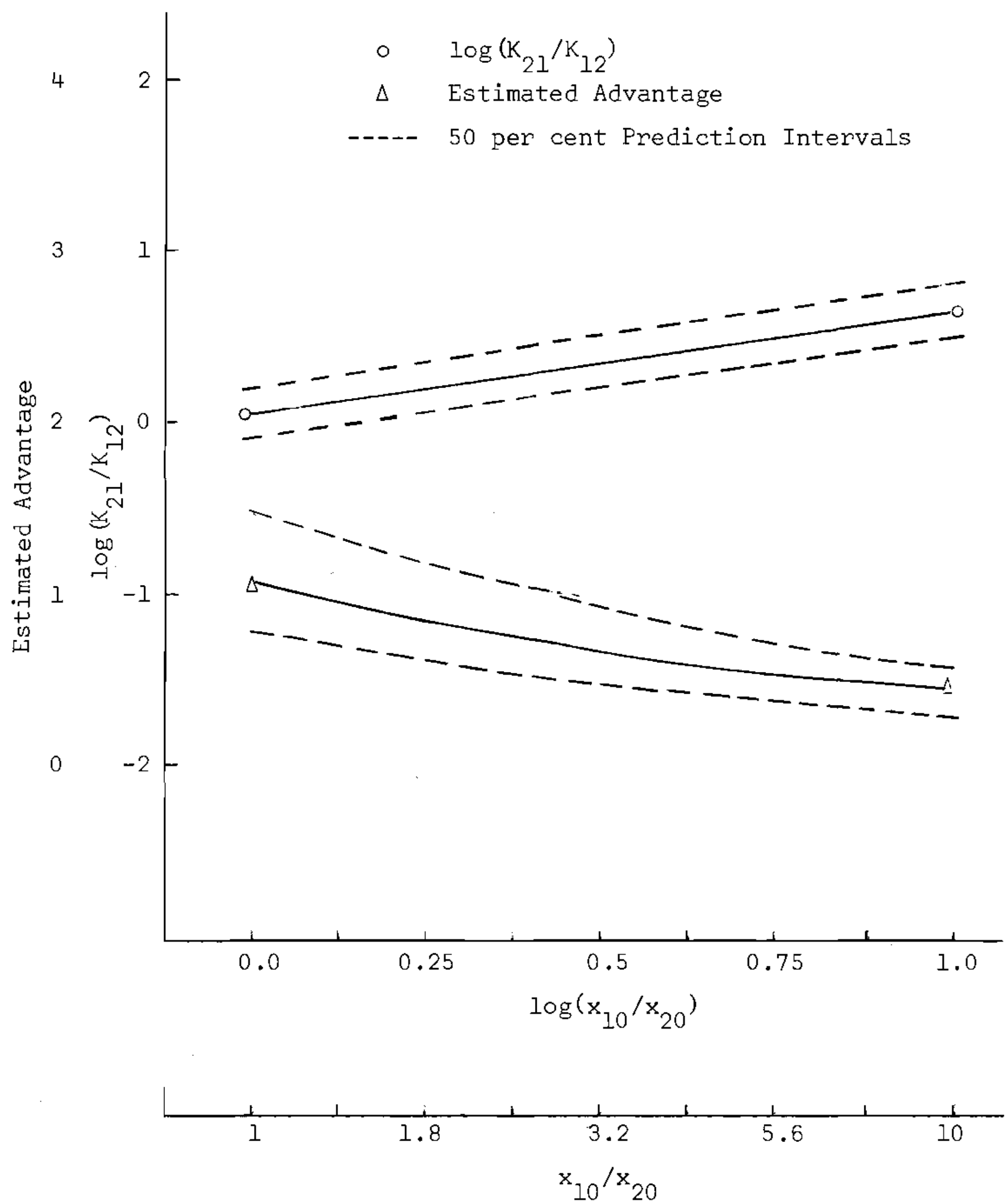

Figure 19. Results of Regression Analysis on Initial Force Model for Category 23

$$C_1 = -0.153$$

and $$C_2 = 0.862.$$

These regression coefficients give

$$\log\,(K_{21}/K_{12}) = -0.153 + 0.862\,\log\,(x_{10}/x_{20}) \tag{48}$$

as the Initial Force Model for estimating the effectiveness ratio, and

$$A = \log^{-1}\,[-0.153 - 0.138\,\log\,(x_{10}/x_{20})] \tag{49}$$

as the Initial Force Model for estimating advantage. This model is graphically illustrated in Figure 20.

The side estimated to have the advantage agreed with the winning side in 65 per cent of the battles studied.

Category 32

This category is defined as that class of battles involving no more than two divisions and with one side suffering casualties exceeding 10 per cent of their initial strength while the other side's casualties are less than or equal to 10 per cent.

The results of the regression analyses were insensitive to the choice of α. With $\alpha = 1$ the following regression coefficients were computed:

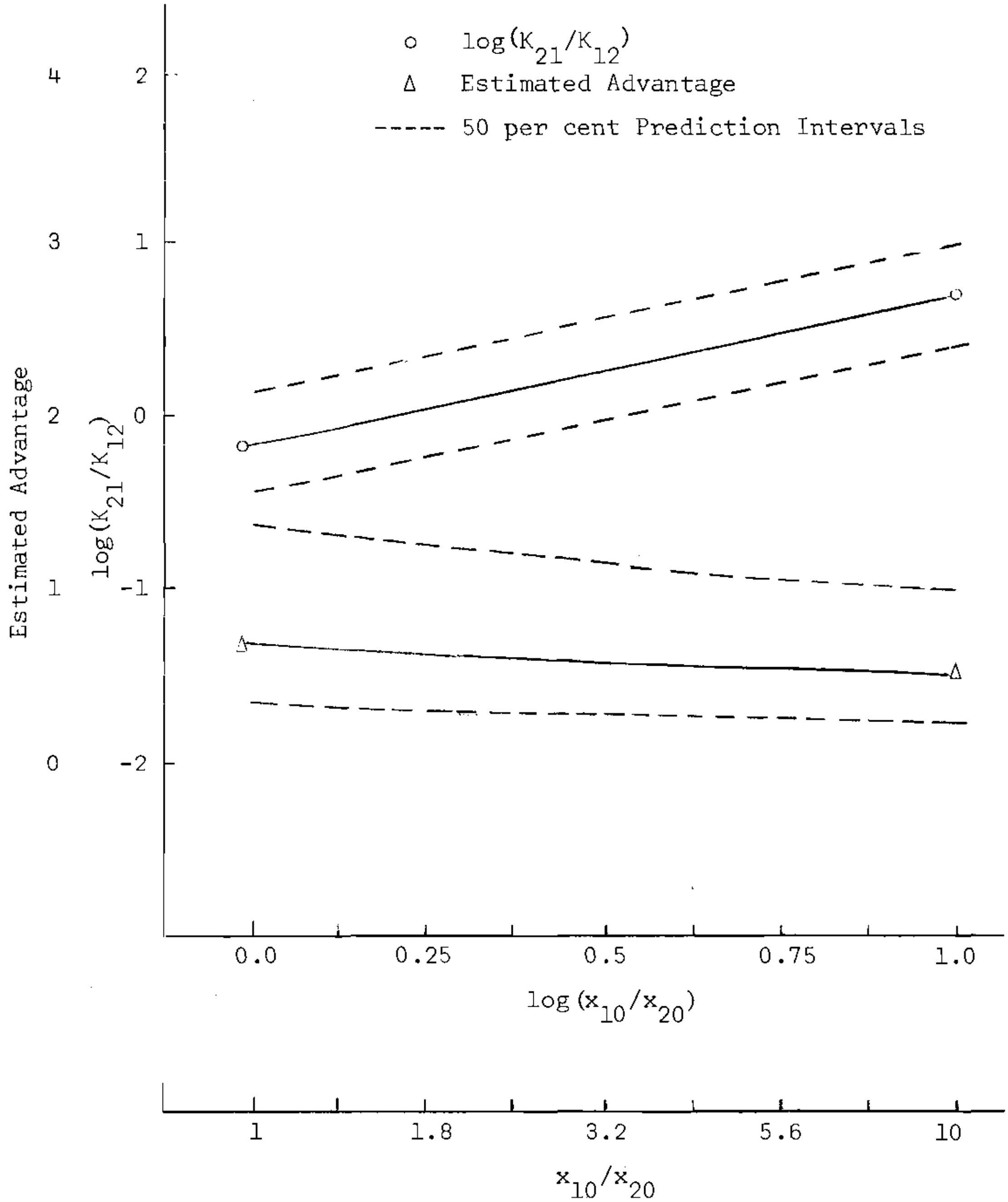

Figure 20. Results of Regression Analysis on Initial Force Model for Category 31

$$C_1 = -0.170$$

and $$C_2 = 0.254 \quad .$$

These regression coefficients give

$$\log (K_{21}/K_{12}) = -0.170 + 0.254 \log (x_{10}/x_{20}) \tag{50}$$

as the Initial Force Model for estimating the effectiveness ratio, and

$$A = \log^{-1} [-0.170 - 0.746 \log (x_{10}/x_{20})] \tag{51}$$

as the Initial Force Model for estimating advantage. This model is graphically illustrated in Figure 21.

As in Categories 12 and 22 the negative value of C_1 indicates that in general the unit effectiveness of the larger force is higher than the inferior force.

The agreement between the side having estimated advantage and the winning side was 70 per cent for this category of combat.

Category 33

This category is the final category in the study and is defined as that class of military engagements involving no more than two divisions and with both sides suffering casualties in excess of 10 per cent.

The results of the regression analyses were insensitive to the choice of α. With $\alpha = 1$ the computed regression coefficients are

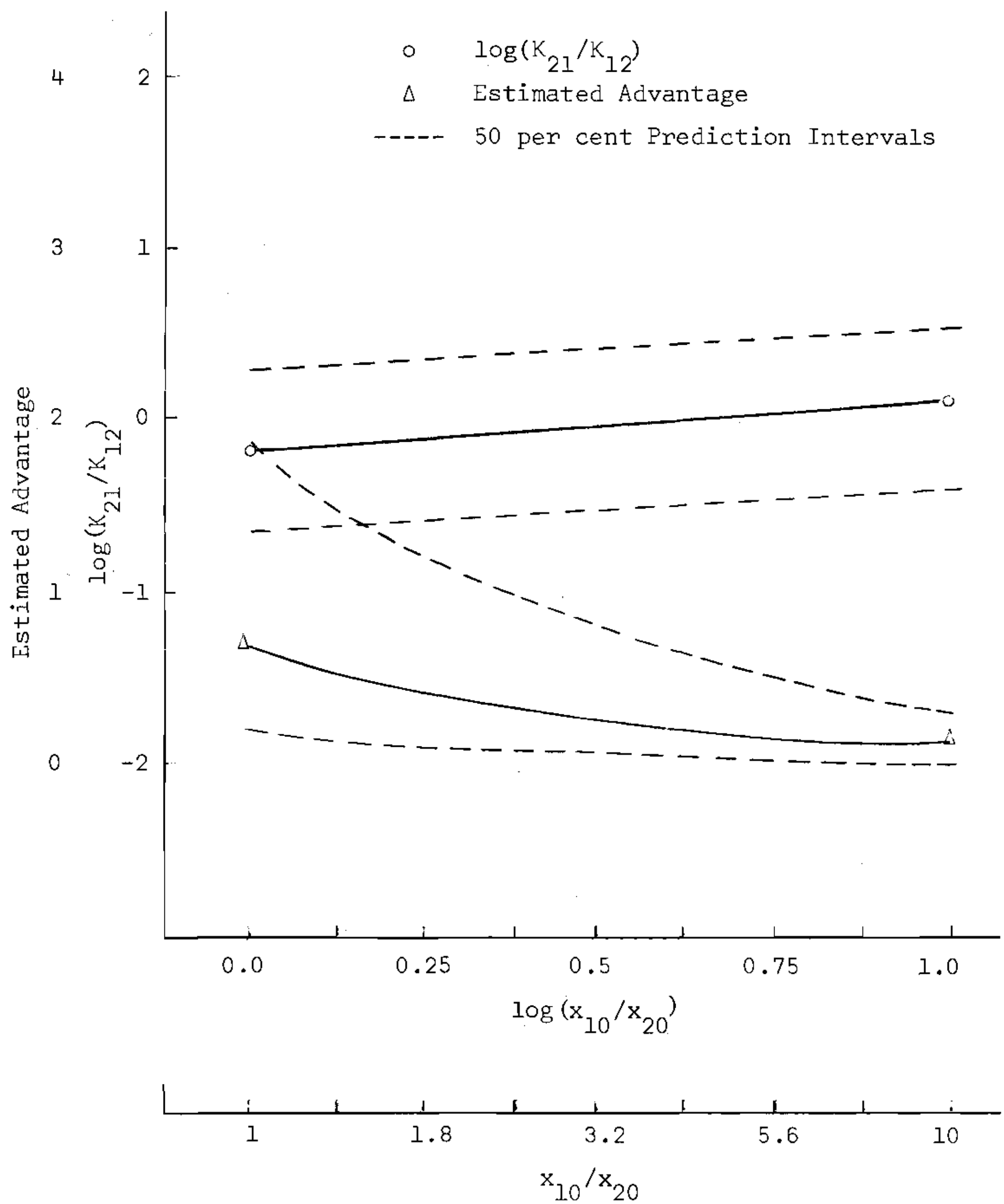

Figure 21. Results of Regression Analysis on Initial Force Model for Category 32

$$C_1 = -0.052$$

and $$C_2 = 0.923 .$$

These regression coefficients give

$$\log (K_{21}/K_{12}) = -0.052 + 0.923 \log (x_{10}/x_{20}) \tag{52}$$

as the Initial Force Model for estimating the effectiveness ratio, and

$$A = \log^{-1} [-0.052 - 0.077 \log (x_{10}/x_{20})] \tag{53}$$

as the Initial Force Model for estimating advantage. This model is graphically illustrated in Figure 22.

The side having estimated advantage agreed with the winning side in 53 per cent of the battles studied or not much better than pure chance.

General Comments

Regression analyses for all categories were insensitive to the choice of α.

Table 9 is a summary of the agreement between the side estimated to have advantage and the winning side.

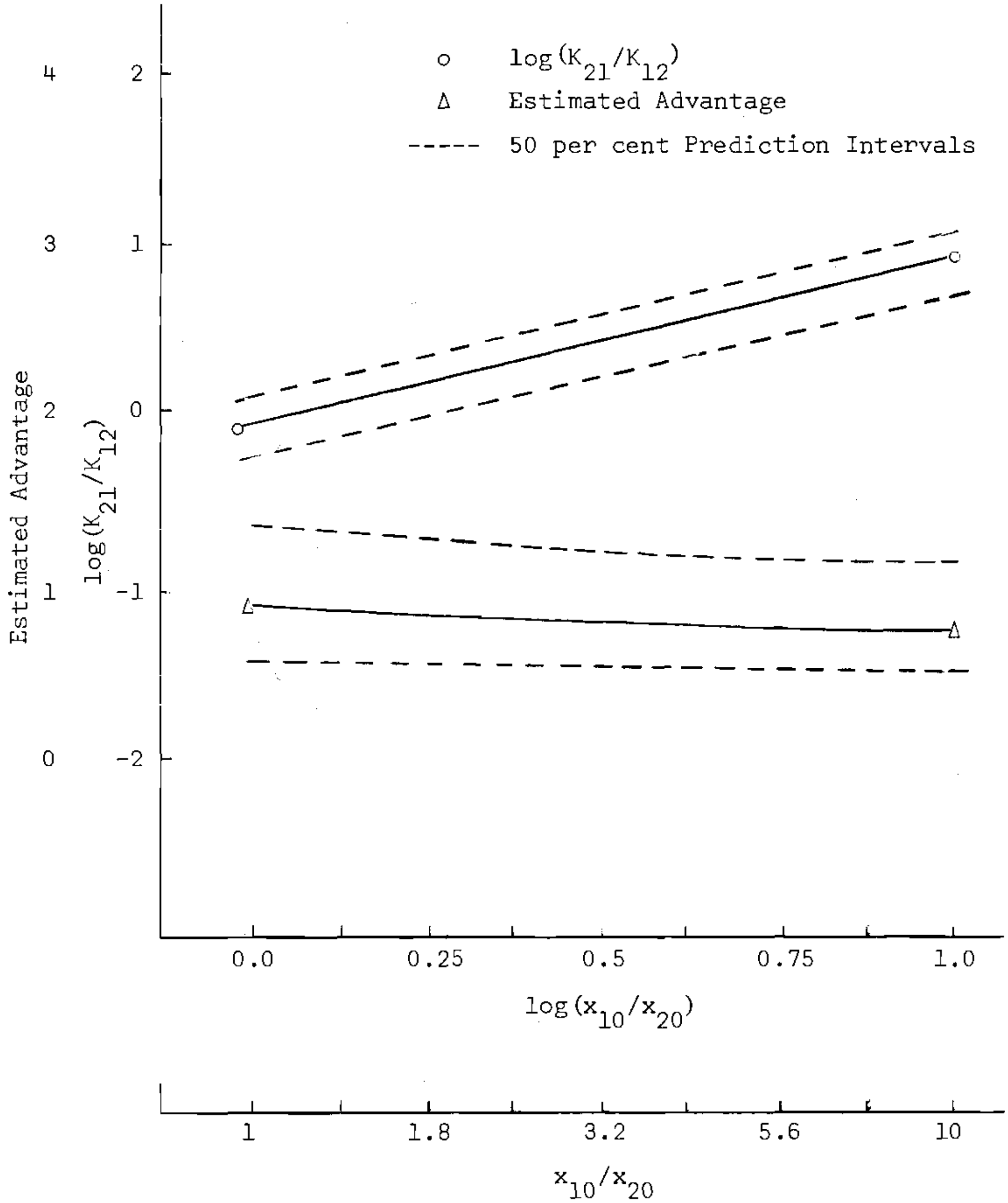

Figure 22. Results of Regression Analysis on Initial Force Model for Category 33

Table 9. Summary of Agreement for Categorized Data

	CASUALTIES					
	Both Sides < 10%		One Side < 10% One Side > 10%		Both Sides > 10%	
Battle Size	Cat.	% Agreement Est. Adv.	Cat.	% Agreement Est. Adv.	Cat.	% Agreement Est. Adv.
> 75,000	11	50	12	70	13	69
75,000 to 30,001	21	56	22	78	23	60
≤ 30,000	31	65	32	70	33	53

An examination of this summary reveals that classification by battle size shows little effect on the model's ability to effectively estimate advantage. However, there is a significant difference in the classification by bitterness of battle or per cent casualties. The models for those categories where one side suffered casualties in excess of 10 per cent while the other side less than or equal to 10 per cent (categories 12, 22, and 32) were more efficient than any of the other six categories studied. The side estimated to have the advantage in each of these categories agreed with the winning side in at least 70 per cent of the battles studied. In each of these categories the regression coefficient C_1 was negative. This would indicate that, in general, for smaller values of force ratio the side with the larger initial strength also possessed the highest unit effectiveness ratio. Therefore, one would expect the side with the greater initial strength to win which is in agreement with any model that depends solely on initial strengths as inputs.

CHAPTER VI

SENSITIVITY ANALYSIS

The results of the regression analyses conducted in Chapter IV and Chapter V were insensitive to the choice of α used in the generalized form of Lanchester's law. In this Chapter explicit solutions to Lanchester's linear and square laws will be investigated for several of the categories studied in Chapter V. Median values will be used for the initial strengths x_{10} and x_{20}, but the effectiveness coefficients, K_{12} and K_{21}, will be allowed to vary over a range of values typical of those found in the category of combat being studied.

To determine the proper range of values for K_{12} and K_{21}, Equations (2') were solved using initial strength and casualty data for a sample of battles in Categories 21, 32, and 33. A value of t equal to one day was used as this value is shown to be the average battle duration for the time period under consideration (see Figure 5). To simplify computation, approximate solutions for K_{12} and K_{21} were obtained by only considering the linear terms in the exponential expansion. These approximate solutions are given by

$$K_{12} = \frac{R_2 x_{20}}{x_{10} x_2} \tag{54}$$

and

$$K_{21} = \frac{R_1 x_{10}}{x_1 x_{20}} , \tag{55}$$

where R_1 and R_2 are the fractional casualties suffered by the "ODD" and "EVEN" sides, respectively, or

$$R_1 = \frac{x_{10} - x_1}{x_{10}} \tag{56}$$

and

$$R_2 = \frac{x_{20} - x_2}{x_{20}} . \tag{57}$$

A Fortran program was written for the IBM 1620 computer to compute the solutions to Lanchester's linear and square laws. A Fortran listing of this program is included in Appendix C. K_{12} and K_{21} were programmed to vary in five equal steps over the range of values determined by the sampling of the three categories studied. A solution was obtained for values of t from t = 0 to t = 2 in steps of 0.1.

A tabulation of the results of these computer runs is given in Tables 10 to 12 for t = 1.0. Five typical runs were selected from each of the three categories and are graphically displayed in Figures 23 to 25.

An examination of Table 10 and Figure 23 reveals that for Category 21 there is no appreciable change in results between Lanchester's linear law and Lanchester's square law regardless of the values of K_{12} and K_{21}. This category, as with Categories 11 and 31, was defined to represent low casualties on both sides and, hence, low effectiveness values. The highest value investigated was 3.89×10^{-6}.

Table 10. Results of Simulation at t = 0 on Category 21

		$K_{12} \times 10^{-6}$									
		1.45		2.06		2.67		3.28		3.89	
LAW	$K_{21}\times10^{-6}$	x_1	x_2	x_1	x_2	x_1	x_2	x_1	x_2	x_1	x_2
	0.95										
L		24.5	21.2	24.5	20.9	24.5	20.6	24.5	20.2	24.5	20.0
S		24.5	21.2	24.5	20.9	24.5	20.5	24.5	20.2	24.5	20.0
	1.41										
L		24.2	21.2	24.2	20.9	24.3	20.6	24.3	20.3	24.3	20.0
S		24.2	21.2	24.2	20.9	24.3	20.6	24.3	20.2	24.3	19.9
	1.87										
L		24.0	21.2	24.0	20.9	24.0	20.6	24.0	20.3	24.0	20.0
S		24.0	21.2	24.0	20.9	24.0	20.6	24.0	20.2	24.0	19.9
	2.33										
L		23.8	21.2	23.8	20.9	23.8	20.6	23.8	20.3	23.8	20.0
S		23.8	21.2	23.8	20.9	23.8	20.6	23.8	20.2	23.8	19.9
	2.79										
L		23.5	21.2	23.5	20.9	23.6	20.6	23.6	20.3	23.6	20.0
S		23.5	21.2	23.5	20.9	23.5	20.6	23.5	20.2	23.5	19.9

Table 11. Results of Simulation at t = 0 on Category 32

		$K_{12} \times 10^{-5}$									
		1.50		2.37		3.25		4.12		5.00	
LAW	$K_{21} x10^{-5}$	x_1	x_2	x_1	x_2	x_1	x_2	x_1	x_2	x_1	x_2
	0.50										
L		9.7	6.0	9.7	5.5	9.7	5.1	9.7	4.7	9.7	4.3
S		9.7	6.0	9.7	5.4	9.7	4.8	9.7	4.2	9.7	3.6
	0.88										
L		9.4	6.1	9.5	5.6	9.5	5.1	9.5	4.7	9.5	4.3
S		9.4	6.0	9.5	5.4	9.5	4.8	9.5	4.2	9.5	3.6
	1.25										
L		9.2	6.1	9.2	5.6	9.3	5.1	9.3	4.7	9.3	4.3
S		9.2	6.0	9.2	5.4	9.3	4.8	9.3	4.2	9.3	3.6
	1.62										
L		9.0	6.1	9.0	5.6	9.1	5.1	9.1	4.7	9.1	4.3
S		8.9	6.0	9.0	5.4	9.0	4.8	9.1	4.3	9.1	3.7
	2.00										
L		8.8	6.1	8.8	5.6	8.9	5.2	8.9	4.7	8.9	4.4
S		8.7	6.0	8.8	5.4	8.8	4.9	8.9	4.3	8.9	3.7

Table 12. Results of Simulation at t = 0 on Category 33

		$K_{12} \times 10^{-5}$									
		2.52		3.79		5.07		6.34		7.62	
LAW	$K_{21} x 10^{-5}$	x_1	x_2	x	x_2	x_1	x_2	x_1	x_2	x_1	x_2
	2.10										
L		7.8	6.5	7.8	5.8	7.9	5.2	7.9	4.7	8.0	4.2
S		7.7	6.3	7.7	5.5	7.8	4.6	7.9	3.7	8.0	2.3
	2.89										
L		7.3	6.5	7.3	5.9	7.5	5.3	7.5	4.8	7.6	4.3
S		7.1	6.4	7.2	5.5	7.4	4.7	7.5	3.9	7.6	3.0
	3.68										
L		6.9	6.6	7.0	5.9	7.1	5.3	7.2	4.8	7.2	4.3
S		6.6	6.4	6.8	5.6	6.9	4.8	7.0	4.0	7.2	3.1
	4.47										
L		6.5	6.6	6.6	6.0	6.7	5.4	6.8	4.9	6.9	4.4
S		6.1	6.5	6.3	5.7	6.4	4.9	6.6	4.1	6.8	3.3
	5.26										
L		6.1	6.6	6.3	6.4	6.4	5.5	6.5	4.9	6.6	4.5
S		5.6	6.5	5.8	6.0	6.0	5.0	6.2	4.2	6.4	3.4

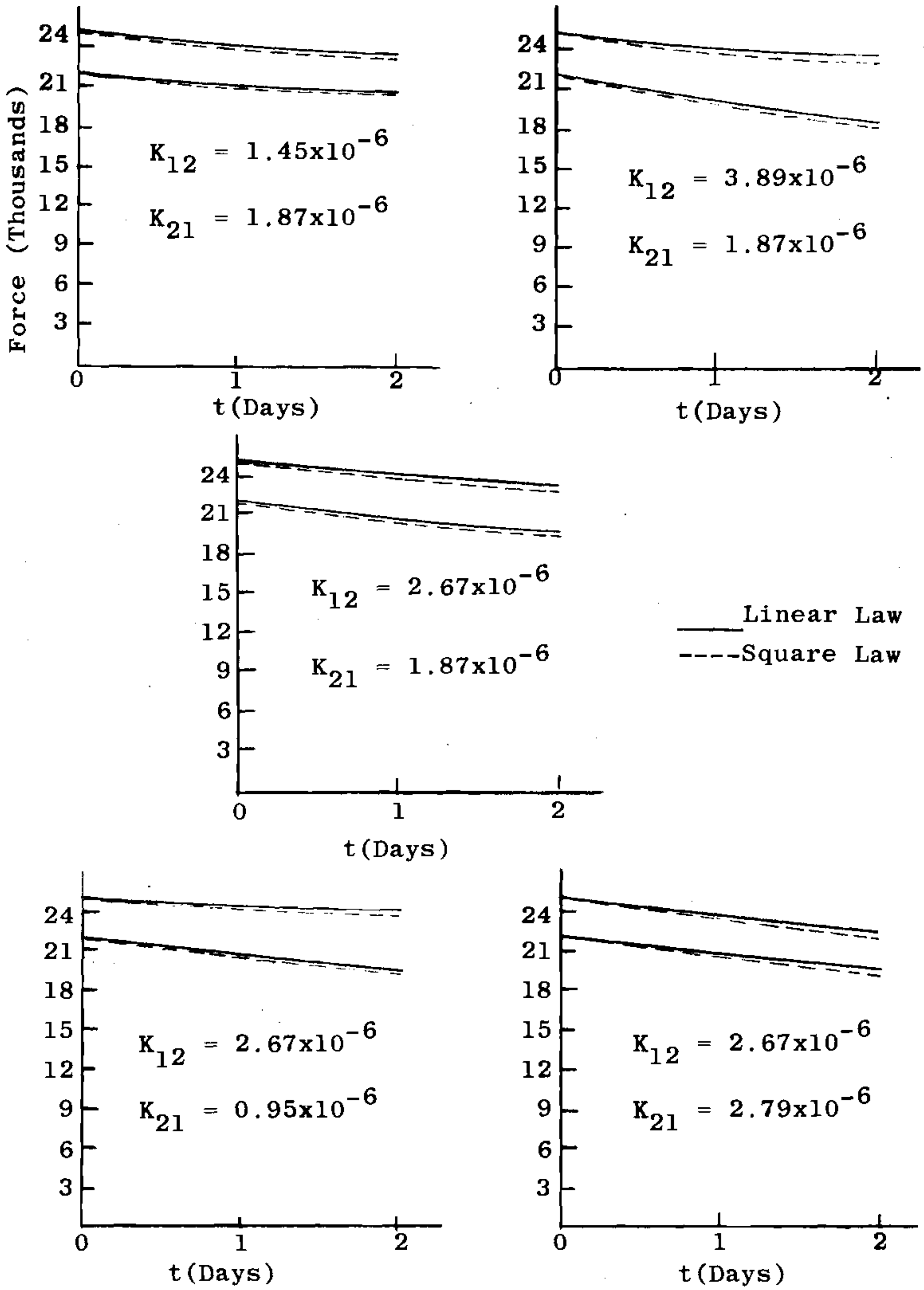

Figure 23. Simulation Results for Category 21

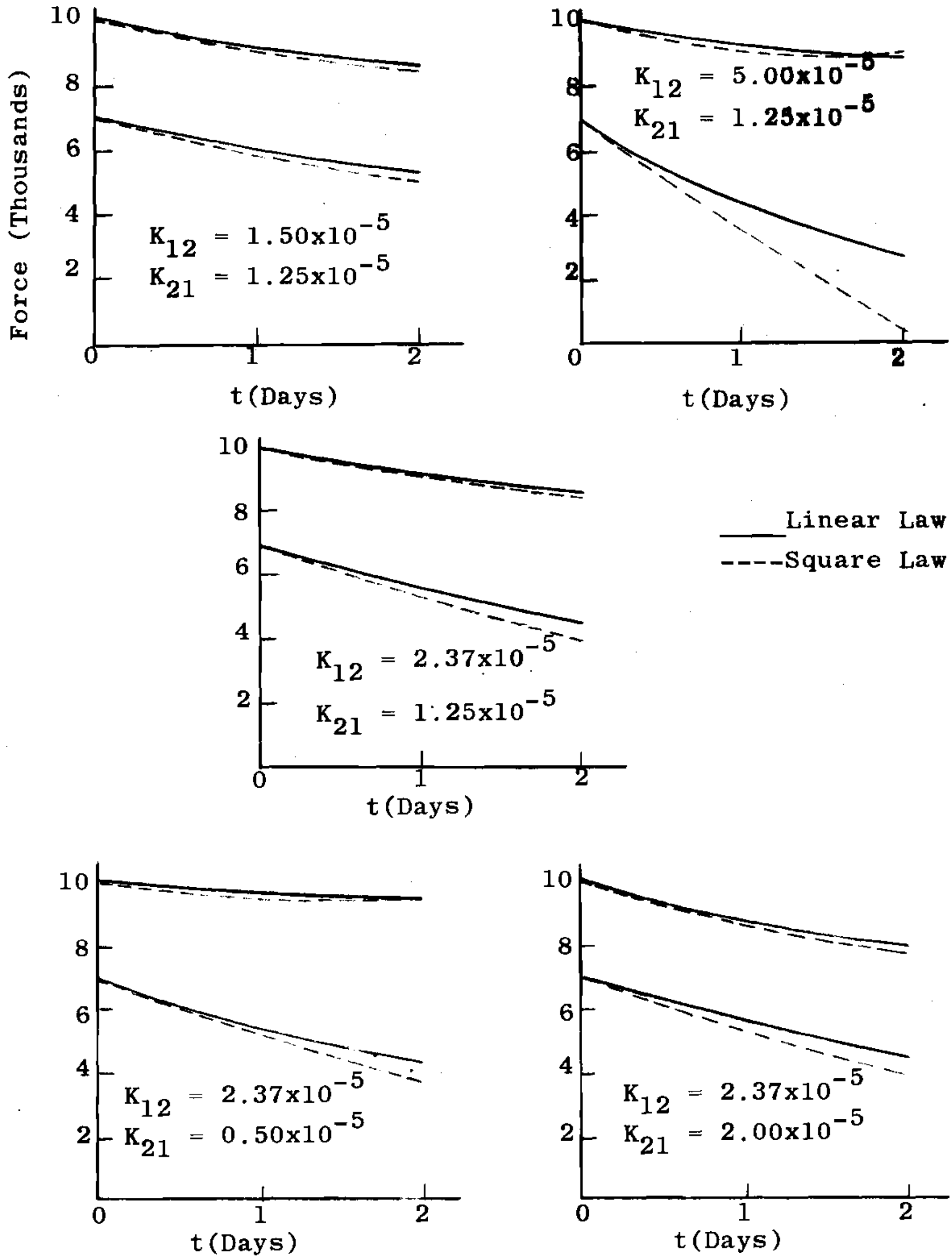

Figure 24. Simulation Results for Category 32

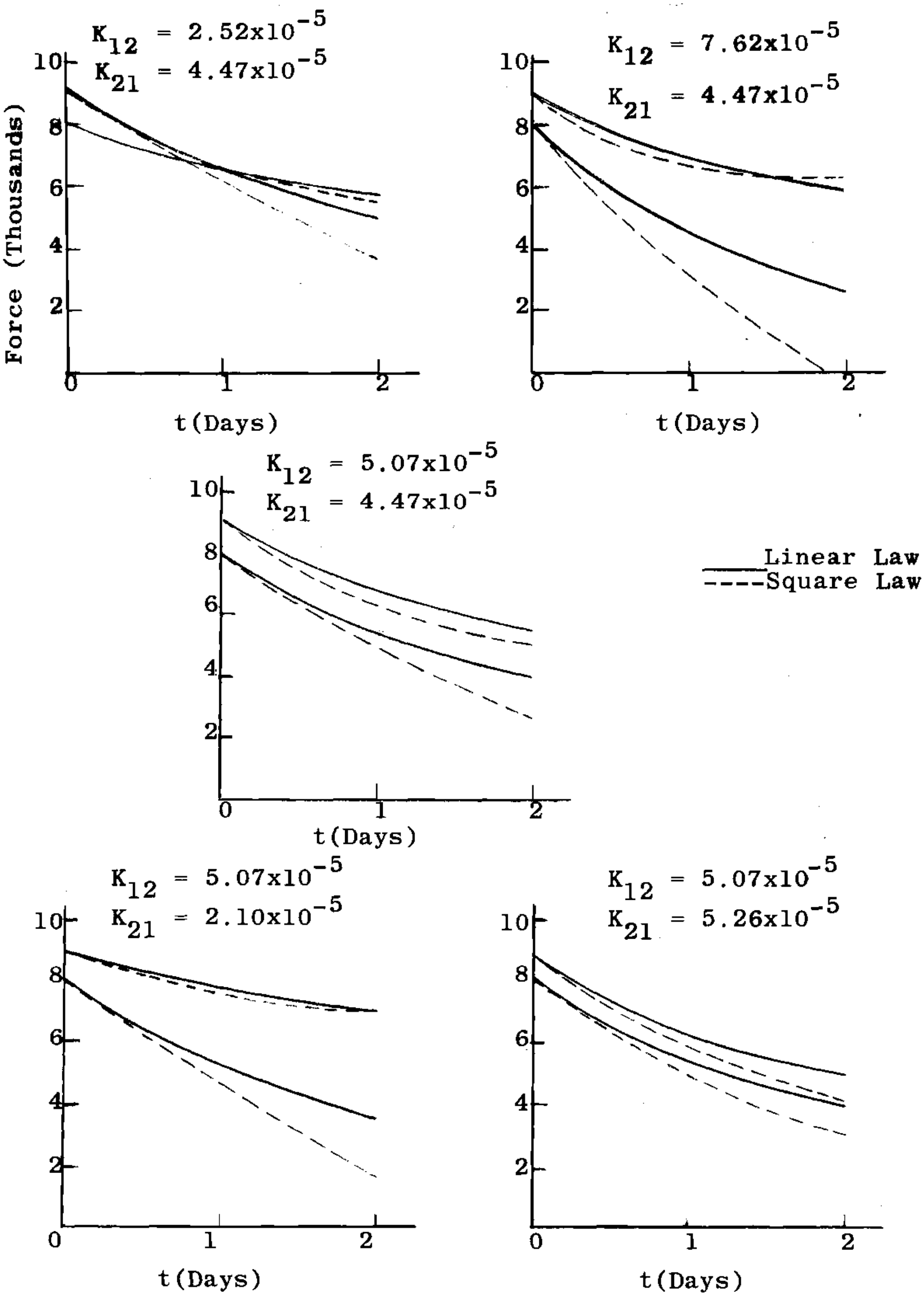

Figure 25. Simulation Results for Category 33

Tables 11 and 12 and Figures 24 and 25 begin to depict appreciable differences in the flow of battle depending upon whether Lanchester's linear law or Lanchester's square law was used in the solution. These differences depend upon the magnitude of the effectiveness coefficients. In general, values of effectiveness coefficients of less than 3.0×10^{-5} resulted in differences in fractional casualties (R_1 or R_2) of less than 4 per cent between the square law and linear law solutions. Differences approached 25 per cent as the effectiveness coefficients increased to 7.6×10^{-5}.

Categories 11, 21, and 31 by definition have low values for effectiveness coefficients. The largest, found in the sampling of the data in Category 21, was only 3.75×10^{-6}. These three categories represent almost 40 per cent of the total battles studied. In addition, many of the battles in Categories 12, 22, and 32 have effectiveness coefficients less than 3×10^{-5}. These facts may in a large part account for the lack of sensitiveness to changes in the exponent α of the generalized Lanchester's law observed in Chapters IV and V.

CHAPTER VII

TIME STABILITY ANALYSIS

Although the two models (Initial Force and Attacker-Defender Models) under consideration in this study do not differ in principle, they do differ as to their input conditions and, as such, may behave differently with time. An analysis to determine the time behavior of the models must therefore treat each model separately. This analysis will determine if these models differ with time and, if so, if any discernible trend exists.

Initial Force Model

Since this model is based on a bivariate regression analysis over an extended period of time, a test will be made to determine if the regression coefficients vary significantly with time. The simplest and most obvious analysis would be a graphical analysis of the regression coefficients with time or battle period. To facilitate this analysis, the data were divided into 11 battle periods of approximately 100 battles each. Although the battles are not uniformly distributed with time, the battle periods give a good time spread in most instances. Points are plotted at the ends of the time periods rather than at the midpoints.

Figures 26 and 27 graphically illustrate the stability of these coefficients with time. Although the coefficients appear to vary considerably, no trend with time is apparent in either C_1 or C_2. The dotted line on the graph represents the value of C in the general model which considered all data.

A t test was conducted to determine if any of the values differed significantly from the mean. Table 13 gives the t values along with the value of the intercept C_1, the expected value of the intercept $E[C_1]$, and the value of $t_{.025;100}$.

Table 13. Test for Significance of the Variation of C_1 with Time (Initial Force Model)

Time	C_1	$E[C_1]$	Computed t	$t_{.025;n}$
1620-1690	-0.085	0.0	1.06	1.98
1690-1712	0.076	0.0	0.93	1.98
1712-1759	-0.020	0.0	0.28	1.98
1759-1793	-0.104	0.0	1.05	1.98
1793-1796	-0.117	0.0	1.63	1.98
1796-1800	-0.094	0.0	1.21	1.98
1800-1810	-0.062	0.0	0.98	1.98
1810-1813	-0.007	0.0	1.59	1.98
1813-1849	0.054	0.0	0.70	1.98
1849-1870	-0.103	0.0	1.64	1.98
1870-1905	-0.005	0.0	0.06	1.99

As in Chapter IV (page 70),

$$t = \frac{C_1 - E[C_1]}{S_{A|B}\sqrt{\frac{1}{n} + \frac{(\bar{B})^2}{\sum_{i=1}^{n} (B_i - \bar{B})^2}}}$$

where

$A = \log (K_{21}/K_{12})$,

$B = \log (x_{10}/x_{20})$.

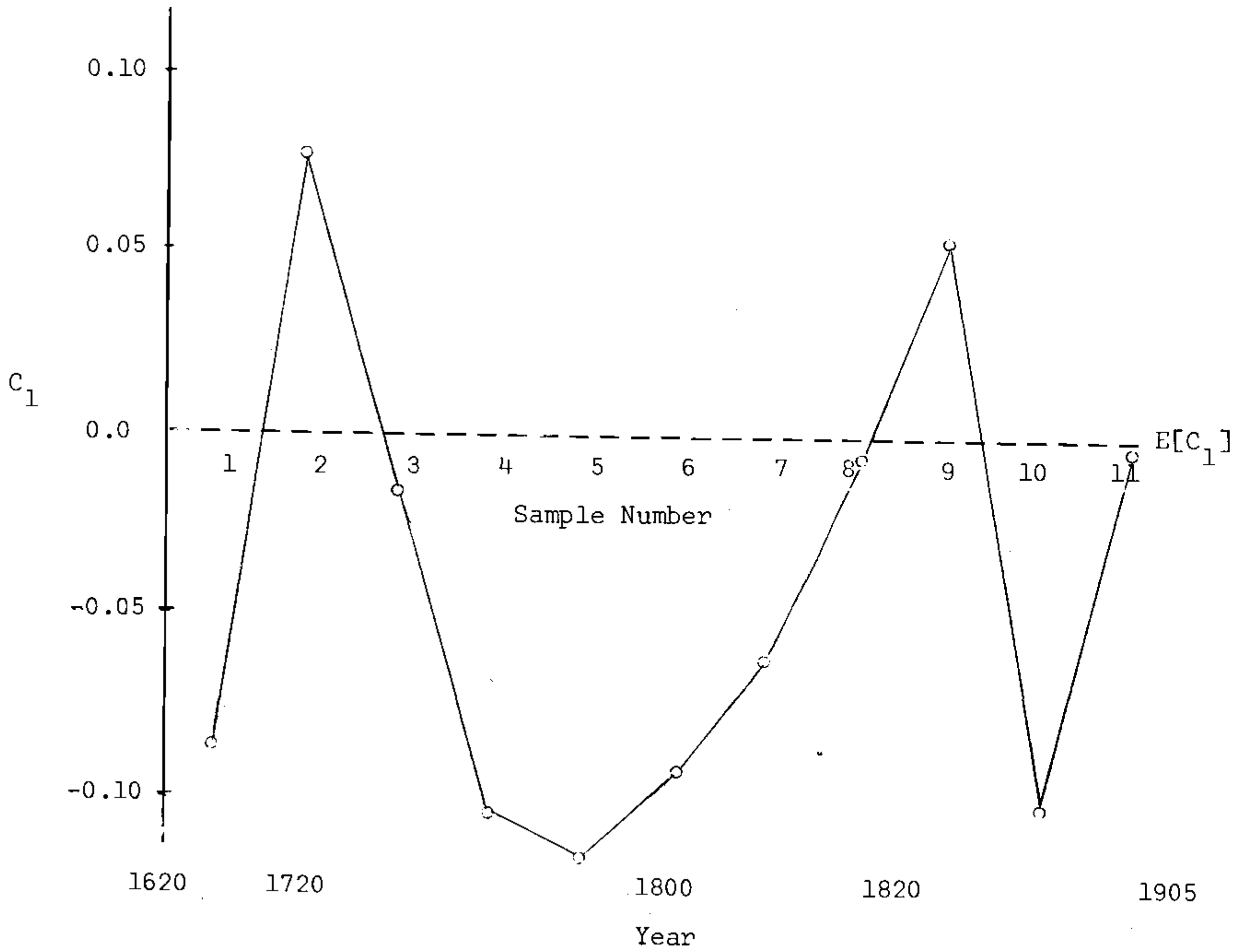

Figure 26. Variation of Intercept C_1 with Time (Initial Force Model)

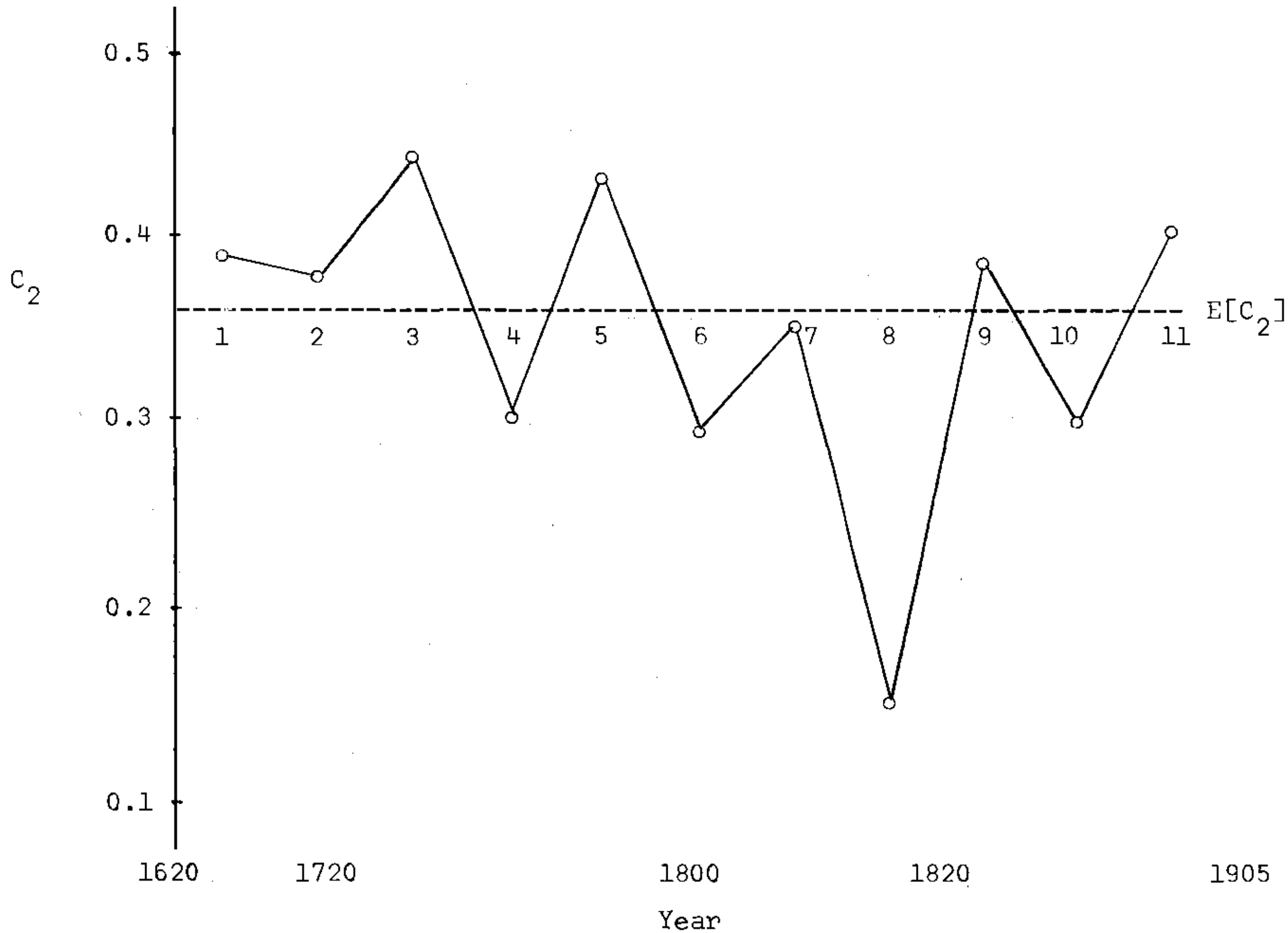

Figure 27. Variation of Slope C_2 with Time (Initial Force Model)

The hypothesis that $C = E[C_1]$ will be rejected whenever $t \geq t_{\alpha/2;n-2}$. An α of 0.05 was used for all tests. Since in Chapter IV it was shown that C_1 did not significantly differ from zero, $E[C_1]$ was taken to be zero. It should be noticed that for the intercept, the computed t value was always well under $t_{.025;100}$ and, therefore, there was no reason to believe that the overall value, or expected value for the intercept coefficient, differed with time. Figure 26 shows the scatter about the mean.

A similar analysis was made for slope C_2. In this analysis

$$t = \frac{(C_2 - E[C_2])\, S_{\log x_{10}/x_{20}} \sqrt{N - 1}}{S_{\log K_{21}/K_{12} | \log x_{10}/x_{20}}}$$

$E[C_2]$ was determined to be 0.361 in the previous chapter. Figure 27 illustrates the scatter about the mean or expected slope, shown by the dotted line. Table 14 again gives the value of the slope, expected value of the slope, the computed t value and the value for $t_{.025;n}$.

Table 14. Test for Significance of the Variation of C_2 with Time (Initial Force Model)

Time	C_2	$E[C_2]$	Computed t	$t_{.025;n}$
1620-1690	0.388	0.361	0.10	1.98
1690-1712	0.379	0.361	0.10	1.98
1712-1759	0.442	0.361	0.48	1.98
1759-1793	0.297	0.361	0.29	1.98
1793-1796	0.433	0.361	0.31	1.98
1796-1800	0.292	0.361	0.28	1.98
1800-1810	0.349	0.361	0.05	1.98
1810-1813	0.146	0.361	1.78	1.98
1813-1849	0.381	0.361	0.09	1.98
1849-1870	0.296	0.361	0.28	1.98
1870-1905	0.402	0.361	0.20	1.99

Attacker-Defender Model

A similar analysis of the stability of the coefficients of regression for the Attacker-Defender Model was performed. The same division of data was used in this analysis as in the stability analysis of the Initial Force Model. A separate bivariate regression was then performed on each division of data. Figures 28 and 29 graphically depict the variation of the regression coefficients with time. Again, no general trend with time is apparent in the variation of either C_1 or C_2.

Table 15 gives the result of a t test performed to determine if any of the variations in C_1 with time are statistically significant.

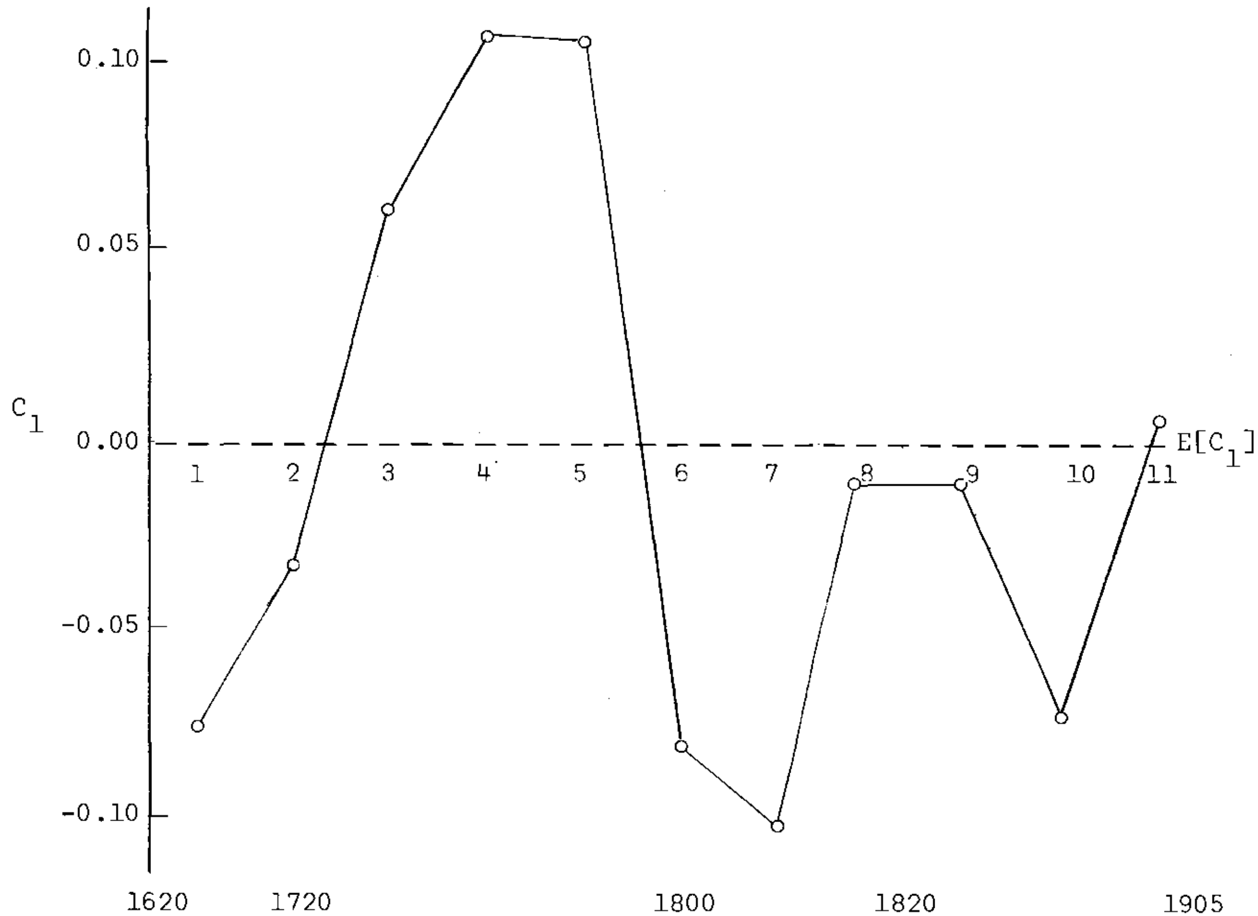

Figure 28. Variation of Intercept C_1 with Time (Attacker-Defender Model)

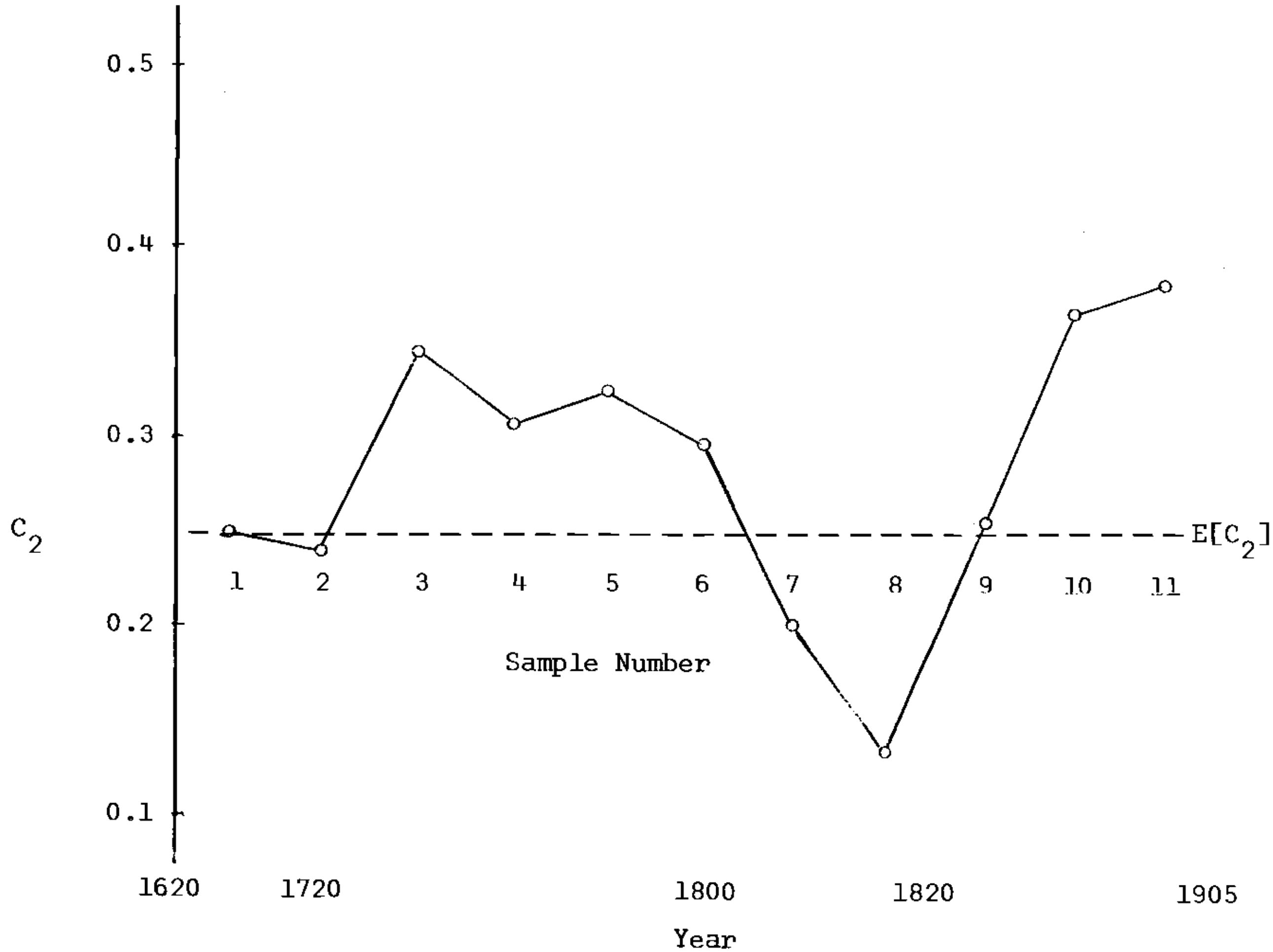

Figure 29. Variation of Slope C_2 with Time (Attacker-Defender Model)

Table 15. Test for Significance of the Variation of C_1 with Time (Attacker-Defender Model)

Time	C_1	$E[C_1]$	Computed t	$t_{.025;n}$
1620-1690	-0.076	0.0	1.28	1.98
1690-1712	-0.037	0.0	0.74	1.98
1712-1759	0.059	0.0	1.15	1.98
1759-1793	0.107	0.0	1.85	1.98
1793-1796	0.104	0.0	1.72	1.98
1796-1800	-0.081	0.0	1.72	1.98
1800-1810	-0.101	0.0	2.51	1.98
1810-1813	-0.012	0.0	0.29	1.98
1813-1849	-0.010	0.0	0.22	1.98
1849-1870	-0.073	0.0	1.74	1.98
1870-1905	0.006	0.0	0.13	1.99

With the exception of the time period between 1800 and 1810, the computed t values are well below the value for the .05 level of significance. No cause could be discerned for the low value of the intercept coefficient, C_1, during this time period. This low value would tend to indicate that the defense position was significantly stronger during this battle period than for the rest of the time under study. The fact that the value of C_1 for the other ten battle periods did not differ significantly from the expected value (the value computed for the combined data) is reason to accept the expected value of C_1 as being stable with time.

A similar analysis was performed to determine if the observed values of the slope (C_2) differed significantly from the expected value $E[C_2]$ obtained from an analysis of all data. Table 16 gives the results of this analysis.

Table 16. Test for Significance of the Variation of C_2 with Time (Attacker-Defender Model)

Time	C_2	$E[C_2]$	Computed t	$t_{.025;n}$
1620-1690	0.248	0.247	0.01	1.98
1690-1712	0.241	0.247	0.06	1.98
1712-1759	0.345	0.247	0.81	1.98
1759-1793	0.308	0.247	0.38	1.98
1793-1796	0.322	0.247	0.46	1.98
1796-1800	0.298	0.247	0.33	1.98
1800-1810	0.198	0.247	0.35	1.98
1810-1813	0.133	0.247	1.24	1.98
1813-1849	0.251	0.247	0.03	1.98
1849-1870	0.367	0.247	0.79	1.98
1870-1905	0.387	0.247	1.10	1.99

Table 16 indicates that none of the samples for the entire time period approaches significance at the .05 level of significance, indicating no reason to doubt the time stability of the expected value of the slope C_2. Furthermore, since the regression coefficients appear to be stable with time, the models based upon these coefficients should be time stable.

It is interesting to compare the value of the effectiveness ratio K_{21}/K_{12}, as estimated with the Attacker-Defender Model ($\alpha = 0$), with the value computed by Engel (21). Engel used actual combat attrition data from the battle of Iwo Jima and computed K_{21}/K_{12} to be 5.1. The Attacker-Defender Model estimates K_{21}/K_{12} to be 4.2. The closeness of this estimate is further evidence of the validity and time stability of this model. In fact, much of the slight discrepancy observed may well have been due to the uncertainty in the initial defender strength.

CHAPTER VIII

DISCUSSION OF RESULTS, CONCLUSIONS AND RECOMMENDATIONS

This chapter will summarize the results of the experiments which were reported in Chapters IV, V, VI, and VII. The intention of the experiments and the results which were achieved will be stated in a concise manner to facilitate the forming of the conclusions and the recommendations of this study.

Discussion

Advantage Parameter

An advantage parameter which made use of terminal battle data was defined by Equation (21). This equation is stated as

$$A(x_{1f},x_{2f}) = \frac{1 - (x_{1f}/x_{10})^{2 - \alpha}}{1 - (x_{2f}/x_{20})^{2 - \alpha}} \tag{21}$$

This advantage parameter was used in the regression analyses in deriving the various models which were obtained in this study. In addition, since the parameter is a measure of the relative advantage of the two sides, the side estimated to have the advantage was compared with the winner of the engagement. If

$$A(x_{1f},x_{2f}) \leq 1$$

"ODD" is estimated to have the advantage, and if

$$A(x_{1f}, x_{2f}) > 1$$

"EVEN" is estimated to have the advantage. The side estimated to have the advantage agreed with the winner in over 79 per cent of the 1,081 battles which were considered in this analysis. The estimation was insensitive to the choice of α used in the advantage parameter. These results closely parallel those obtained by Willard in 1962. Willard performed an analysis of the same data, but he used a stochastic model which had been developed by Brown. With this model, Willard correctly estimated the winner in 76.4 per cent of the battles when $\alpha = 0$, and 77.8 per cent of the battles were correctly predicted when $\alpha = 1$. Willard was unable to obtain this accuracy when he attempted to use a deterministic model similar to the advantage parameter of this study.

Models

Three basic models are developed in this study: a Control Model where "ODD" is defined as the side winning the conflict, an Initial Force Model where "ODD" is defined as the side having initially the superior numerical strength, and an Attacker-Defender Model where "ODD" is defined as the side determined to be the aggressor in the engagement.

The Control Model. The Control Model was developed to determine whether battles have followed Lanchester's square law ($\alpha = 0$), Lanchester's linear law ($\alpha = 1$), or some other generalized Lanchester law. α is the exponent in Lanchester's generalized equations

$$\dot{x}_1 = -K_{21}x_{10}\left(\frac{x_1}{x_{10}}\right)^{\alpha} x_2 \tag{16}$$

$$\dot{x}_2 = -K_{12}x_{20}\left(\frac{x_2}{x_{20}}\right)^{\alpha} x_1$$

Analyses were performed with values of α varying from $\alpha = 2$ to $\alpha = -1$.

Bivariate regression analysis techniques were used to establish a relationship between the ratio of initial strengths and the effectiveness ratio K_{21}/K_{12} of the form

$$\log (K_{21}/K_{12}) = C_1 + C_2 \log (x_{10}/x_{20}) \tag{23}$$

This relationship was then used to develop a model for estimating the advantage. This model had the form

$$A = \log^{-1} [C_1 + (C_2 - 1) \log (x_{10}/x_{20})] . \tag{25}$$

"ODD" was estimated to have the advantage when $A \leq 1$, and "EVEN" when $A > 1$.

The results of the regression analysis indicated that the model was insensitive to the choice of α. $\alpha = 1$ resulted in the following equations for estimating effectiveness ratio and advantage:

$$\log (K_{21}/K_{12}) = -0.283 + 0.484 \log (x_{10}/x_{20}) \tag{26}$$

and

$$A = \log^{-1} [-0.283 - 0.516 \log (x_{10}/x_{20})] . \qquad (27)$$

Initial Force Model. The Initial Force Model depends upon only the initial force ratio as input data.

Bivariate regression analyses of Bodart's battle data again were insensitive to the value of α chosen. These analyses resulted in

$$\alpha = 1 ,$$

$$C_1 = -0.05 ,$$

and $$C_2 = 0.361 .$$

Based upon these results, the Initial Force Model for estimating the effectiveness ratio becomes

$$\log (K_{21}/K_{12}) = -0.05 + 0.361 \log (x_{10}/x_{20}) , \qquad (28)$$

and for estimating advantage becomes

$$A = \log^{-1} [-0.05 - 0.639 \log (x_{10}/x_{20})] . \qquad (29)$$

A variation of the t test was used to determine if the intercept coefficient, C_1, was significantly different from zero. The results were negative, and Equations (28) and (29) become simply

$$\log (K_{21}/K_{12}) = 0.361 \log (x_{10}/x_{20}) , \tag{30}$$

and

$$A = \log^{-1} [-0.639 \log (x_{10}/x_{20})] . \tag{31}$$

The side estimated to have the advantage with only the initial force ratio used as inputs agreed with the winner of an engagement in 66 per cent of the battles studied, in contrast to the 79 per cent agreement achieved by the advantage parameter. The advantage parameter used the additional data as to the number of casualties on each side at the termination of the conflict, and was expected to achieve a greater percentage of agreement.

The prediction interval about the estimated advantage parameter was also employed to determine the probability of the numerically inferior force having the advantage. An analysis that was based upon these prediction intervals revealed that a side outnumbered by a factor of three to one still has a 25 per cent chance of possessing advantage. These results are in agreement with the results achieved by Robinson (50) in an analog study of a combination Lanchester-type linear-square law model. In this study, Robinson analyzed the utility of a model which allows one side to have a casualty rate depicted by Lanchester's square law and the other side to have a casualty rate which does not follow either the square law or the linear law. This model, referred to as Robinson's Model III* is indecisive for initial force ratios of

$$0.30 < x_{10}/x_{20} < 3.3 \ .$$

Attacker-Defender Model. The Attacker-Defender Model is similar to the other two models defined, except in this model, "ODD" is defined as the side determined to be the aggressor.

Bivariate regression analyses were again performed to determine the form of the law (value of α) and the appropriate regression coefficient C_1 and C_2. Similar to the other two models studied, this model was insensitive to the choice of α. Therefore, $\alpha = 1$ was arbitrarily chosen to be representative of the general case. This value of α resulted in the following regression coefficients:

$$C_1 = -0.007 \ ,$$

$$\text{and} \qquad C_2 = \ 0.247 \ .$$

Again, due to the small magnitude of C_1, a t test was used to determine

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*Robinson's Model III can be expressed as

$$\dot{x}_1 = -Cx_{10}x_2 - Dk_{21}x_2 \ ,$$

$$\dot{x}_2 = -k_{12}x_1$$

where C and D are coefficients used to vary the analog model.

if this coefficient was significantly different from zero. The results of the test were negative, and the Attacker-Defender model for estimating effectiveness ratio becomes

$$\log (K_{21}/K_{12}) = -0.247 \log (x_{10}/x_{20}) \;, \tag{34}$$

and for estimating advantage becomes

$$A = \log^{-1} [-0.753 \log (x_{10}/x_{20})] \;. \tag{35}$$

The Attacker-Defender Model estimated that on the average, the side with superior numbers had the advantage regardless of whether he is the attacker or the defender.

Categorized Combat Situations

Many researchers have stated the desirability of categorizing military combat into identifiable classes. In Chapter III, the author defined nine categories or types of combat based upon the total initial numerical strengths of the two opposing sides and the fractional casualties.

Regression analyses were used to develop models for estimating effectiveness ratio and advantage for each category. Results of the regression analyses revealed that all categories were insensitive to the choice of α or the form of Lanchester's law used. Classification by battle size had little effect on the model's ability to estimate advantage. However, there is a significant difference in the model's ability to estimate advantage under the per cent casualty classification.

For those categories where one side suffered 10 per cent or less casualties while the other side suffered greater than 10 per cent casualties (Categories 12, 22, and 32), the agreement between estimated advantage and the winning side averaged approximately 73 per cent. For the other two casualty breakdowns (both sides less than or equal to 10 per cent and both sides greater than 10 per cent), the per cent agreement was only 57 per cent and 59 per cent.

Sensitivity Analysis

An analysis was conducted to determine how the sensitivity to the form of the law (Lanchester's square law or Lanchester's linear law) was affected by the magnitude of the effectiveness coefficients K_{12} and K_{21}. This analysis indicated that if the effectiveness coefficients were less than 3×10^{-5}, differences in per cent casualties of less than 4 per cent between the square law and linear law solutions can be expected. As the magnitude of the effectiveness coefficients increased so did the differences. Casualty differences between linear law and square law solutions approached 25 per cent as the magnitude of the effectiveness coefficients approached 8×10^{-5}.

Time Stability of Models

Bodart's battle data were separated into 11 battle periods of approximately 100 battles each. A separate regression analysis was performed for each group and for each model. The variability of the regression coefficients was studied, both through a graphical analysis and with statistical procedures, to determine if the coefficients varied significantly with time and, if so, if this variation could be predicted.

The graphical analysis revealed no apparent trends in the regression coefficients of either model with time. A t test was performed to determine if the variations of the regression coefficients between the battle period data and the composite data were statistically significant. This test gave no reason to believe that the regression coefficients were unstable or varied significantly with time. Therefore, it was concluded that the models studied are time stable.

Conclusions

Using Bodart's *Lexicon* (covering 1,081 land battles fought between 1620 and 1905) as a data source, the following conclusions were obtained:

(1) Regression analyses of the above data indicated that Lanchester's generalized law is insensitive to the value of the exponent α. No difference could be determined between Lanchester's square and linear laws.

(2) The empirical relationship between the effectiveness ratio and the initial strength ratio for the aggregate battle data was determined to be $\log (K_{21}/K_{12}) = 0.361 \log (x_{10}/x_{20})$.

(3) The advantage parameter $A(x_1,x_2)$ agreed with the winner of the engagement in 79 per cent of the battles studied. This result is in good agreement with previous research based upon stochastic analyses.

(4) Simulation results, using empirically determined effectiveness coefficients, indicated that the difference in casualties predicted by Lanchester's square law and Lanchester's linear law is less than 4 per cent for values of effectiveness coefficients less than 3×10^{-5}. Values of effectiveness coefficients below this magnitude were encoun-

tered in over 70 per cent of the battles studied. These simulation results support the first conclusion which was based upon regression analyses.

(5) A two-way classification of battles by battle size and per cent casualties revealed that the classification by battle size resulted in no improvement in the models' ability to estimate advantage. Classification by per cent casualties resulted in no improvement for those classes in which both sides had either light or heavy losses. However, for the classes in which one side had light losses and the other heavy losses, an average improvement of 7 per cent over the unclassified data was obtained.

(6) Analyses of battles in which one side had heavy losses and the other side light losses indicate that, on the average, the larger force has the higher effectiveness coefficient when ratios of initial strengths are less than three to one. For larger initial strength ratios, however, there is a trend for the smaller force to have the higher effectiveness coefficient. In general, the logarithm of the effectiveness ratio is inversely proportional to the logarithm of the initial strength ratio.

(7) The models developed proved to be time invariant over the time period under consideration.

(8) The distribution of battles by battle size follows an exponential distribution of the form $f(x) = \lambda e^{-\lambda x}$, where x is the number of combatants in thousands. For the data considered in this study it was found that $\lambda = 0.025$.

Recommendations

The following areas of additional research are recommended:

(1) This study has shown that models based upon the initial strengths of the opposing sides and the identification of the aggressor can provide an estimate of the average effectiveness ratio expected as a function of these parameters, but they are insufficient to estimate such a ratio for a specific battle situation. What is needed is more explicit research into the nature of the coefficients of effectiveness, K_{21} and K_{12}, with particular emphasis on the offensive and defensive attributes of the opposing sides. Such factors as types of weapons employed, types of terrain, communications, and logistics could well be used to define these attributes.

(2) Another area for further research could well be the effect on coefficients of effectiveness of such intangibles as morale of troops. Rashevsky (49) developed a theory for analyzing the effect of morale which involves a critical ratio or threshold of active to passive troops to continue the conflict. This theory maintains that when the ratio of actives to passives drops below the threshold necessary to maintain the influence of the actives over the passive, then that group for which this occurs ceases fighting, even though physically able to continue. This may help to explain the fact that, in the data of this study, 50 per cent of all battles were decided before either side lost more than 12.5 per cent of its total force. If sufficient details on troop morale could be obtained on a large number of past battles, the study of the relationship between morale and effectiveness ratio would be a very worthwhile research project.

(3) A final recommended area for research is a study of past military histories directed toward the determination of an empirical relationships between motion of the front lines or forward edge of the battle area (FEBA) and the effectiveness ratio. Weiss (57) presented a theoretical treatment of this problem in which he assumed that each commander predetermines the casualty rate that he is willing to accept, and advances and retreats according to whether the actual casualty rate is less than, or greater than, the predetermined rate. A limited sample of combat data appeared to support his analysis, but to determine its validity would require a detailed research of a rather large magnitude. This research would yield additional insight into methods for predicting the course of a given engagement.

APPENDIX A

EQUATION DEVELOPMENT

Equality of Fighting Strength

Proof of equality of fighting strength is based upon the equation

$$\dot{x}_1/\dot{x}_2 = x_1/x_2 \ . \qquad \text{(A-1)}$$

When the percentage rate of loss of a given side is equal to the percentage rate of loss of its opponents, the two sides are equal in fighting strength and if fought to near annihilation the combat would result in a draw.

The model exemplified by Equation (8) in the text is

$$\dot{x}_1 = -k_{21}x_2x_1/m_1 - k_{41}x_{40}m_1 \qquad \text{(8)}$$

$$\dot{x}_2 = -k_{12}x_1x_2/m_2 - k_{32}x_{30}m_2 \ .$$

Substituting these values into Equation (A-1) results in

$$\frac{k_{21}x_2}{m_1} + \frac{k_{41}x_{40}m_1}{x_1} = \frac{k_{12}x_1}{m_2} + \frac{k_{32}x_{30}m_2}{x_2} \ , \qquad \text{(A-2)}$$

and clearing fractions

$$k_{21}x_2^2x_1m_2 + k_{41}x_{40}x_2m_1^2m_2 = k_{12}x_1^2x_2m_1 + k_{32}x_1x_{30}m_1m_2^2 \ . \tag{A-3}$$

According to Equation (10) in the text

$$m_1^2 = \frac{k_{21}x_2x_1}{k_{41}x_{40}} \ , \qquad m_2^2 = \frac{k_{12}x_1x_2}{k_{32}x_{30}} \ . \tag{10}$$

Substituting these values for m_1^2 and m_2^2 in Equation (A-3) yields

$$k_{21}x_2^2x_1m_2 = k_{12}x_1^2x_2m_1 \ . \tag{A-4}$$

Both sides of Equation (A-4) were squared and the values for m_1^2 and m_2^2 given in Equation (10) were substituted. This resulted in

$$k_{21}x_2^2(k_{41}x_{40}) = k_{12}x_1^2(k_{32}x_{30}) \ ,$$

which is the desired result.

Regression Line Prediction Interval

The regression line represents the mean or expected value of log (K_{21}/K_{12}) given log (x_{10}/x_{20}). The confidence interval is merely a determination of how well one can determine this mean. It is often the case, as in this study, that a statement about the mean is unimportant, whereas a probability statement about a specific observation is relevant. To simplify the presentation, the regression line will be stated simply as

$$\tilde{y} = a + bx \ .$$

Bowker and Lieberman have a relatively rigorous proof that the prediction interval for y^o given x^o is

$$a + bx^o \pm t_{\alpha/2;n-2} S_{y|x} \sqrt{1 + \frac{1}{n} + \frac{(x^o - \bar{x})^2}{\sum_{i=1}^{n} (x_i - \bar{x})^2}} \qquad \text{(A-5)}$$

if y^o is normally distributed with mean $a + bx^o$ and variance σ^2. The probability is then $1 - \alpha$ that a future observation y^o corresponding to x^o will lie in this interval. $S_{y|x}$ is an estimate of the variability about the regression line and is defined as

$$S_{y|x} = \sqrt{\frac{\sum_{i=1}^{n} (y_i - \tilde{y}_i)^2}{n - 2}} \qquad \text{(A-6)}$$

It is apparent that $\sum_{i=1}^{n} (y_i - \tilde{y}_i)^2$ is the sum of squares of the deviations about the fitted line. The prediction interval defined in Chapter IV is merely Equation (A-5) where

$$x = \log (x_{10}/x_{20}) \ ,$$

$$y = \log (K_{21}/K_{12}) \ ,$$

$$a = C_1 ,$$

$$\text{and} \qquad b = C_2 .$$

The proof of the correctness of this interval would follow the same reasoning given in Bowker and Lieberman, pages 254 and 255. Now it only remains to be shown that log (K_{21}/K_{12}) is normally distributed.

The values of log (K_{21}/K_{12}) for the Initial Force Model were analyzed as to their range of values, and it was observed that more than 65 per cent of these values were within the interval

$$-0.50 \leq \log (K_{21}/K_{12}) \leq 0.50 .$$

The mean and standard deviation of log (K_{21}/K_{12}) were computed, and, based upon the hypothesized normal distribution, ten subintervals were selected. The observed frequencies in each of these subintervals were tabulated, and the accumulated frequency was plotted on normal probability paper. This graphical analysis is shown in Figure (A-1). The closeness of the plotted points to a straight line tends to indicate that the distribution is approximately normal with a mean of zero and a standard deviation of approximately 0.5.

The χ^2 test for goodness of fit was applied to these data to determine if the hypothesis that the sample data has a normal distribution with $\mu_x = 0$ and $\sigma_x = 0.5$ should be accepted or rejected. Since both the mean and the standard deviation were estimated, two degrees of freedom were lost. Thus, the total number of degrees of freedom for this test is

$$(k-1) - 2 = k - 3 = 7$$

where k is the number of subintervals. The chi squared statistic was computed as

$$\chi^2 = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i}$$

where O_i is the observed frequency in each subinterval and E_i is the theoretical frequency based upon the hypothesized normal distribution.

Table A-1 shows the results of this test. Since the observed value of $\chi^2 = 10.61$ is smaller than 11.07, we accept, at the 5 per cent level, the hypothesis that the distribution is normal.

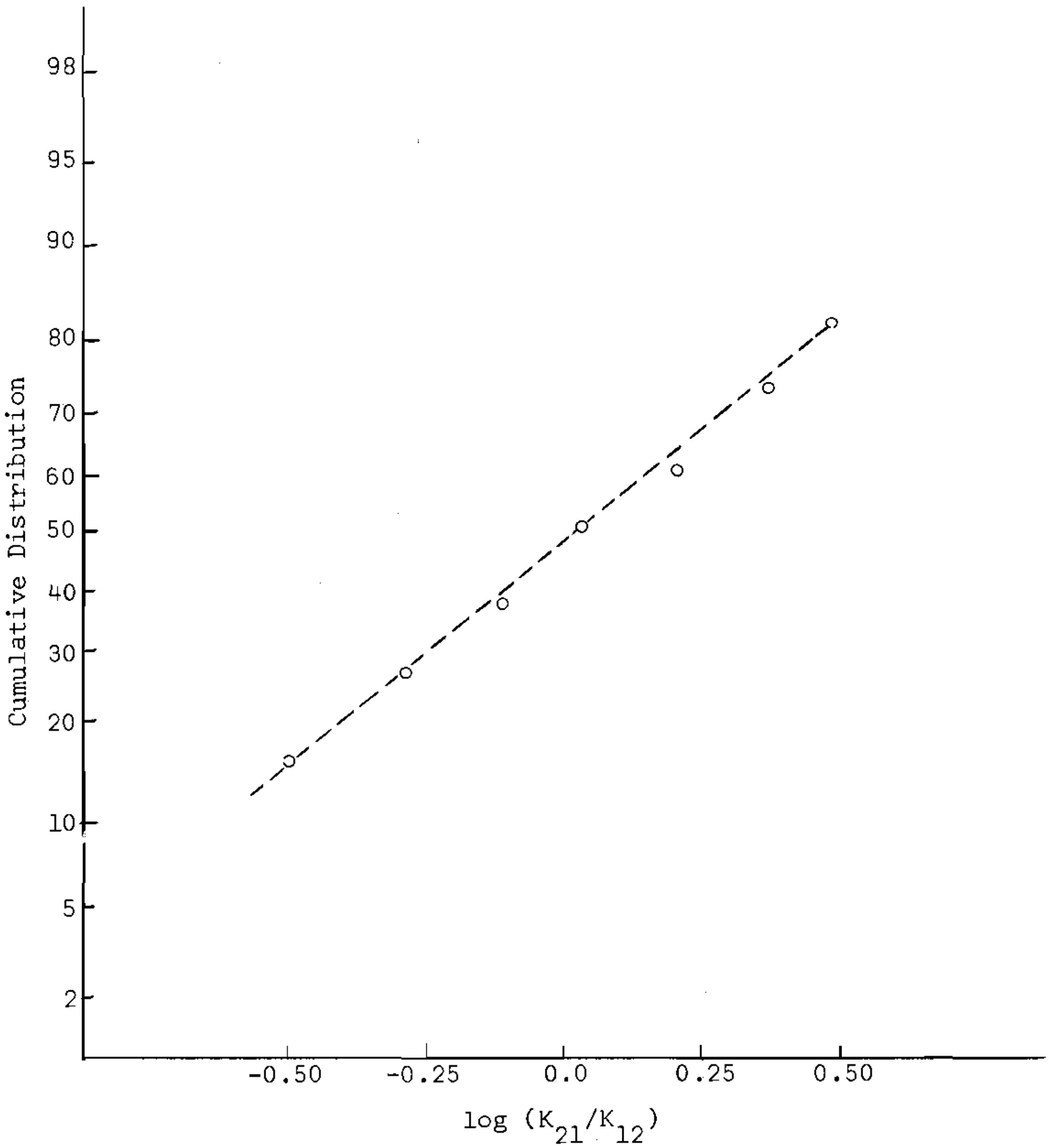

Figure A-1. Graphical Analysis of the Distribution of Log (K_{21}/K_{12})

Table A-1. Chi Square Test for Normality of Log (K_{21}/K_{12})

Interval	S from Mean at End Points	Observed Frequency	Theoretical Frequency	$\frac{(O-E)^2}{E}$
< -0.500	< -1.00	172	172	0.00
-0.50 to -0.29	-0.58	116	132	1.94
-0.29 to -0.117	-0.23	120	137	2.11
-0.117 to +0.045	+0.09	142	140	0.03
+0.045 to +0.208	+0.42	133	135	0.03
+0.208 to +0.382	+0.76	121	126	0.20
+0.382 to +0.500	+1.00	77	70	0.71
> +0.500	> +1.00	203	172	5.59
		1084	1084	10.61

$\overline{\log K_{21}/K_{12}} = 0.0$

$\chi^2_{.05;5} = 11.070$

$S_{\log(K_{21}/K_{12})} = 0.5$

APPENDIX B

DATA LISTING

This Appendix contains the IBM card input format, a listing of participants as identified by Bodart (5), and a complete listing of all data used in this study.

CARD FORMAT BODART DECK

1 2 3	4	5 6	7 8	9-15	16	17-22	23	24-29	30 31	32 33	34-40	41	42-47	48	49-54	55	57	59	60-62
Page # (Bodart)	Item #	Winner A	Winner B	Winners Initial Strength	No Data	Winners Casualties	No Data	Winners Missing + Captured	Loser A	Loser B	Loser's Initial Strength	No Data	Loser's Casualties	No Data	Losers Missing + Captured	Code 1	Code 2	Code 3	Year

Fields 16, 23, 41, 48 have an X-PCH if there was no information as opposed to no known casualties.

Field 55 Code 1: 1 Initial Strengths only or asymmetric casualty data
2 Dead + Wounded both sides
3 End of category
4 End of deck

Code 2: 1 Belagerung
2 Erstürmüng = Cat II
3 Einnehmen
4 Kapitulation
0 Treffen, Gefecht, Schlacht = Cat I

Code 3: 1 Battle won by the Attacker
2 Battle won by the Defender

All Fields: Right-justified.

CODE FOR PARTICIPANTS

1. France
2. Austria
3. Great Britain
4. Russia
5. Prussia
6. Spain
7. Turkey
8. Holland
9. Sweden
10. United States of America
11. Denmark
12. Italy
13. Japan
14. Poland
15. Portugal
16. Confederate States of America
17. Hungary = 36
18. Naples = 48
19. Egypt
20. Chile (Government)
21. China
22. Greece
23. Serbia
24. Bulgaria
25. Tyrol
26. Bavaria
27. Württenberg
28.
29.
30. Kaiserlich (Holy Roman Empire)
31. German Insurgents
32. Cossacks
33. Tartars
34. Germany
35. Bohemia
36. Parliament (English Civil War) = 17
37. Saxony
38. Royalist (English Civil War) = 41
39. "Frondeurs"
40. Brandenburg
41. "Koniglich" = 38
42. Venice
43. Ireland
44. Piedmont
45. "Auslandischen"
46. Sardinia
47. Modena
48. Naples = 18
49. Hesse
50. Scotland
51. Geneva
52. "Reichstruppen"
53. Hanover
54. Vende'
55. Jurpfalz
56. Pfaltz
57. Vatican
58. Morocco
59. Mexico
60. Garibaldi's Troops
61. Roumania
62. Chile (Congress)
63. Boers
64. Makin
65. Tarawa
66. Kwajalein
67. Eniwetok
68. Saipan
69. Guam
70. Tiniam
71. Pelelui
72. Iwo Jima
73. Iwo Jima*
74. Okinawa
75. Okinawa*
76. No. Okinawa
77. Ie Shima

Card Cols. (5,6) or (7,8) or (30,31) or (32,33).

Table B-1. Listing of Bodart Battle Data, Category 11

188204	60000	100-	07	100000	2000-	2		2739
106430	60000	100-	07	42000	2000-	2		1685
199105	80000	150-	02	14900	1400	135002		1744
51130204	37000	200-	17	45000	500	60002		1849
269301	52000	300-	05	35000	200-	2		2792
305102	42000	300-	01	60000	700	10002		2795
396302	40000	300-	01	39000	500	19002		2809
246334	57000	500	30001	25000	1500	30002		2762
107330	50000	500-	07	80000	2000	5002		1686
497314	50000	500-	04	26000	1000	45002		1831
27330203	60000	600-	01	30000	1500-	2		1793
579104	70000	800-	07	35000	3000-	2		1877
500319	43000	500-	07	33000	1000	100002		1839
45910405	160000	2000-	01	100000	3000	20002		2813
50840204	51000	700	20017	50000	1300	2002		1849
306102	43000	600-	01	37000	1100	5002		2795
29040208	28000	400	25001	60000	3000	24002		2794
505202	55000	900	45046	75000	900	3002		1848
298201	88000	1500-	02	77000	3000	8002		1794
28620308	60000	1000-	01	60000	2500-	2		1794
055330	60000	1000-	0937	46000	2000-	2		1632
39830126	60000	1000-	02	46000	2800	24002		1809
28720308	90000	1500-	01	90000	7000-	2		2794
39820126	52000	1000-	02	58000	2750	40002		1809
20430601	50000	1000-	4602	30000	1000	15002		1745
250104	30000	600-	0733	60000	3000-	2		2769
24910252	50000	1000	40005	30000	1600-	2		2762
391201	35000	700-	06	45000	5000	30002		1808
217202	50000	1100-	05	63000	3000-	2	1	2757
26510204	27000	600-	07	50000	5000	1002		1789
564234	86000	2000-	01	64000	3000	180002		1870
507317	53000	1200-	02	37000	2000-	2		2849
311401	50000	1200	120002	48000	1100	5002		1796
538110	72000	1700-	16	48000	4800-	2		1864
44720405	24000	600	001	50000	1200	10002		2813
244234	52000	1300	20001	100000	3000	20002		2761
562334	40000	1000-	01	60000	2200	18002		1870
561301	60000	1500	30034	20000	1000	8002		1870
541110	60000	1500	20016	30000	1700	16002		1865
400101	72000	2000-	02	78000	2000	60002		1809
268304	36000	1000-	07	64000	3000	10002		1791
537116	60000	1700-	10	100000	11000	18002		2864
560234	200000	6000-	01	180000	6000-	2	4	1870
570134	23000	700-	01	83000	4000	5002		1871
536410	80000	2400-	16	40000	3000-	2		1864
559134	40000	1300	50001	36000	1200	1002		1870
47720204	100000	3500-	01	50000	4000	25002		2814
555234	78000	2800	30001	96000	3600-	2		1870

Table B-1. Listing of Bodart Battle Data, Category 11
(Continued)

352301	84000	3000-	02	72000	3000	40002		1800
210101	110000	4000-	0302	75000	7000	30002		1746
51620301	50000	1800-	04	74000	7300	15002		1855
576204	55000	2000-	07	30000	3000	120002		1877
5823 7	47000	1800	22	35000	500	2		1897
580224	50000	1900-	23	32000	1100-	2		1885
218201	60000	2350	15003	38000	1250	2502		1757
428104	20000	800-	07	60000	1500	20002		1811
514107	20000	800-	04	60000	1600-	2	1	2854
336202	55000	2200	120001	45000	1300	3002		2799
569134	45000	1800	40001	135000	4000	5002		1871
108230	50000	2000-	07	60000	8000	20002		1887
420315	32000	1300-	01	58000	4500-	2		2810
46020204	60000	2500	50001	50000	1500	10002		2813
281302	43000	1800-	01	50000	2000	10002		1793
571134	240000	10000	200001	400000	16000	80002	1	1870
536110	93000	4000	50016	52000	1500	30002		1864
289301	70000	3000-	0302	74000	4000	15002		1794
424315	35000	1500	30001	45000	2700-	2		1811
39920126	66000	3000-	02	74000	6000	50002		1809
357201	55000	2500-	0226	57000	5500	100002		1800
528216	68000	3150	25010	54000	3700	5002		1862
568334	72000	3500	50001	88000	6000	200002		1871
47020204	123000	6000-	01	41000	3000	30002		2814
555134	66000	3400	20001	59000	5500-	2		1870
08340940	19000	1000-	14	70000	3000-	2		1556
410101	74000	4000-	02	60000	3200	30002		1809
45211506	90000	4900	30001	60000	6000	8002		2813
399101	36000	2000-	02	50000	450	25502		1809
310101	36000	2000	40002	45000	1300	13002		1796
43610114	180000	10000-	04	120000	6000-	2		1812
507202	41000	2300	100046	59000	3000	54002		1849
353101	52000	3000-	02	48000	2400	16002		1800
322101	52000	3000-	02	34000	2700	20002		1797
46421506	90000	5300	20001	50000	3200	13002		1813
272402	43000	2600	40001	41000	3000	10002		2793
529116	78000	4700	70010	117000	11000	20002		2862
414301	33000	2000-	06	50000	4000	140002		1809
359101	66000	4000-	02	50000	4100	43002		1800
329302	46000	2900	310001	38000	2000	20002		2799
252304	32000	2000-	0733	57000	4000	50002		2771
154103	90000	6000-	0106	80000	6000	80002		1708
121307	60000	4000-	30	50000	4000-	2		2696
563134	25000	1700-	01	65000	5000	5002		1870
543102	75000	5200	280012	90000	3400	41002		1866
474215	45000	3200	10001	35000	3000	7002		1814
23123403	38000	2700	20001	52000	4900	22002		2759

Table B-1. Listing of Bodart Battle Data, Category 11
(Continued)

173130	63000	4500-	07	60000	6000-	2	1716
29310208	41000	3000-	01	73000	2000-	2	2794
497104	52000	3800	20014	44000	2900	6002	2831
569234	33000	2500	10001	47000	3500	90002	1871
536310	52000	4000-	16	38000	3000-	2	1864
519201	48000	3900	70002	62000	5700	45002	2859
14710308	60000	5000-	0106	62000	2000	60002	2706
553234	58000	4800	20001	84000	3200	6002	1870
533210	65000	5600	40016	45000	2600	44002	1863
331102	46000	4000	200001	41000	3500	45002	2799
562334	40000	3500-	01	90000	3500	15002	1870
397301	45000	4000-	02	40000	3000	20002	1809
397102	39000	3600-	01	41000	3000	50002	2809
564134	38000	3600	70001	52000	4500	25002	1870
455101	100000	10000-	0308	200000	15000	250002	1813
103201	40000	4000-	0806	40000	4000-	2	2678

Table B-1. Listing of Bodart Battle Data, Category 12

04923006	47500	400-	35	32000	6000-	2		1620
312302	46000	400-	01	34000	12000	8002		2796
538310	60000	700-	16	30000	4600-	2		1864
57820461	120000	2000-	07	30000	5000-	2		1877
47410402	90000	1600	30001	25000	2600	5002		2814
27340203	53000	1000-	01	27000	3000	3002		1793
094214	50000	1000-	07	80000	20000-	2		2673
418201	70000	1500-	06	5500	1500	40002	1	1810
444104	90000	2000-	01	50000	6000	120002		2812
251304	40000	1000-	0733	150000	20000-	2		2770
213201	80000	2000	0802	13000	2000	110002	1	1748
47910204	100000	3000-	01	30000	3400	8002		1814
528302	50000	1550	65001	26000	4000	6002		2799
382304	63000	2500-	01	17000	1800	16002		2807
548105	220000	8900	3000237	215000	23600	207002		1866
556134	200000	8300	70001	120000	17000	210002		1870
122230	50000	2100-	07	100000	30000-	2		1697
453201	160000	8000-	0405	100000	16000-	2		1813
45420504	80000	4000-	01	60000	12000	180002		2813
264104	90000	4800-	07	14000	10000	4002	2	2788
540210	55000	3100-	16	38000	15000-	2		1864
45811506	90000	5000	30001	4800	1900	12002	1	1813
538210	70000	4000-	16	40000	8500-	2		1864
25010302	50000	3000-	01	45000	5500	5002		2794
112101	50000	3000	30000830	38000	6000	50002		1690
293401	81000	5000-	0208	46000	5000-	2		1794
372101	96000	6000-	0537	54000	12000	150002		1806
46721506	70000	4600	50001	50000	5400	6002		2813
10411430	76000	5000-	07	107000	15000	50002		2683
22120252	80000	5300	50005	30000	9000	60002		1757
08813001	30000	2000-	07	60000	8000-	2		1664
230304	72000	4800-	05	28000	7000	22002		1759
185104	60000	4000-	07	22000	17000	30002	2	2737
51420107	65000	4400-	04	35000	5000	8002		1854
202105	70000	4600-	0237	75000	9600	56002		2745
33210204	52000	3800	120001	28000	4000	70002		2799
208102	40000	3000	5000106	44000	7000	30002		2746
18030146	53000	4000-	30	37000	6000-	2		1734
119101	40000	3000-	3008	36000	9000	20002		1693
228202	65000	5400	230005	42000	7900	24002		2758
181101	117000	10000-	30	4500	1000	35002	1	1734
301101	35000	3000-	06	50000	10000-	2		1794
579304	67000	6000-	07	29000	4500	225002		1878
48020204	100000	9000-	01	42000	7000	23002		1814
201201	60000	5600	4000308	50000	9000	30002		1745
159104	54000	5000-	09	26000	9200	28002		2709
463101	60000	6000	40002602	40000	5000	50002		1813
188107	80000	8000-	02	40000	5200	4002		2739

Table B-1. Listing of Bodart Battle Data, Category 12 (Continued)

~~118101~~	~~80000~~	~~8000~~	~~10000308~~	~~50000~~	~~12000~~	~~20002~~	~~2693~~
175130	50000	5400-	07	150000	15000	50002	1717
45720308	80000	9000	20000137	70000	6500	135002	2813
565134	30000	3500	10001	60000	5000	40002	1870
576107	40000	5000-	04	60000	4000-	2	2677
530316	80000	10600	200010	130000	11400	60002	2863
~~476201~~	~~40000~~	~~5600-~~	~~0405~~	~~100000~~	~~5000-~~	~~2~~	~~2814~~
535210	100000	18200	230016	50000	5000	40002	1864
465101	37000	6000-	0204	45000	4000-	2 4	1813
524216	39000	6500	50010	51000	4400	6002	2862
363101	65000	12500-	0405	95000	9000-	2	1807
525116	94000	19000	100010	106000	10000	60002	1862

Table B-1. Listing of Bodart Battle Data, Category 13

52020146	143000	14600	280002	130000	13100	87002		2859
211101	98000	10000	10000308	82000	9000	20002		1747
354302	24000	2500	350001	12000	4000-	2	4	1800
554234	187000	19700	50001	113000	12800-	2		1870
45620308	103000	11000	100001	37000	9000	80002		2813
219104	55000	6000-	05	25000	4100	5002		1757
41020306	54000	6000	70001	47000	7100	2002		2809
48121506	60000	6700-	01	52000	4000-	2		1814
552234	82000	9300	140001	41000	8000	123002		1870
432215	46000	5200-	01	42000	10000	70002		2812
598113	145000	17000-	04	210000	46000-	2		1904
217102	54000	6400	160005	33000	8600	54002		2757
095201	50000	6000	40000806	70000	8600	54002		2674
532210	75000	9000-	16	28000	7000	210002	1	1863
393301	50000	6000-	05	30000	18000	120002	1	1808
535110	118000	14300	340016	62000	7000	3002		1864
117101	57000	7000-	0308	63000	6600	14002		2692
497204	80000	10000	50014	50000	9000	5002		2831
171101	80000	10000-	30	9300	3600	57002	1	1713
450201	167000	21200	38000405	97000	11000-	2		1813
597213	135000	17500-	04	150000	16500-	2		1904
599213	314000	41000	004	310000	71000	200002		1905
449101	144000	19200	28000405	93000	12000-	2		1813
498304	78000	10600-	14	37000	7800	30002	2	2831
527310	85000	11700	80016	50000	9400	18002		1862
383201	87000	12000-	04	61000	20000-	2		1807
34030204	50000	7000	200001	35000	7000	40002		2799
17233704	70000	10000-	09	12000	6000	60001	1	1715
45311506	55000	8000	100001	60000	11000	40002		2813
369301	65000	10000-	0402	85000	16000	200002		1805
48720204	120000	19000	400001	72000	25000	170002		2815
115130	50000	8000-	07	100000	20000-	2		1691
486201	71000	12000-	05	84000	16000	80002		1815
523310	60000	10200	280016	40000	9300	10002		2862
222105	35000	6200-	0252	85000	10000	170002		2757
409201	160000	30000	700002	130000	19000	70002		1809
526116	50000	9400	10010	70000	10200	43002		1862
216205	64000	12600	180002	61000	9200	44002		1757
552110	90000	17700	530016	68000	15300	67002		2863
175230	100000	20000-	07	30000	5000	250002	1	1717
575107	35000	7000-	0461	95000	16000-	2		1877
529210	45000	9600	400016	38000	9300	11002		1863
405102	99000	21500	150001	66000	23000	20002		2809
12040308	80000	18000-	01	13000	8000-	2	1	2694
533116	70000	15800	200010	57000	11500	50002		1863
438101	124000	28000-	04	122000	45000	90002		1812
243105	44000	10000	40002	66000	9000	70002		2760
200301	70000	16000-	02	7000	2200	48002	1	1744

Table B-1. Listing of Bodart Battle Data, Category 13 (Continued)

23210402	70000	16000	30005	48000	18700	20002		1759
46110204	325000	75000	500001	175000	45000	150002		2813
599113	40000	9400-	04	58000	13000-	2		1905
554134	63000	14900	110001	113000	17000-	2		1870
137208	50000	12500	5000126	52000	14000	140002		1704
373101	27300	7100-	05	50000	10000	30002		1806
16020308	93000	25000-	01	90000	11000	30002		1709
380201	75000	22000	10000405	83000	23000	30002		1807
445201	33000	10000	1000004	87000	8000	20002		1812
51710301	175000	54000-	04	120000	103000-	2	1	1854
541210	120000	39000	2100016	67000	40000-	2	1	1864
227105	33000	11000	200004	52000	18000	30002		2758
104230	10000	5000	170007	200000	48000-	2	1	2683
598213	140000	75000	500004	40000	27500	125002	1	1904
091407	130000	108000-	42	20000	16000-	2	1	1667

Table B-1. Listing of Bodart Battle Data, Category 21

496304	19000	100-	07	35000	1000	5002	1829
496404	18000	100-	07	20000	500	12002	1829
403115	27000	150-	01	13000	600	18002	2809
586163	25000	180-	03	9500	320	10002	2899
501201	10000	70-	58	30000	2000-	2	1844
521246	30000	200-	57	6000	500	6002	1860
126101	25000	200-	08	25000	400	3002	1702
551105	25000	210-	02	20000	490-	2	1866
530116	25000	200-	10	33000	1200	6002	2863
13043008	24000	200-	01	35000	3000-	2	2703
546105	41000	350-	02	29000	920	7802	1866
124330	22000	200-	0144	38000	3600	2002	1701
549305	26000	250-	02	34000	600	8002	1866
312201	40000	400-	02	24000	600	12002	1796
321201	18000	200-	02	17000	300	35002	1797
308402	36000	400-	01	11000	500-	2	2796
51110204	25000	300	017	38000	500	32002	1349
320301	43000	500-	02	24000	700-	2	1797
341303	17000	200-	0108	21000	2000-	2	2799
441304	30000	400-	01	30000	400	8002	1812
543205	10000	130-	02	23000	370	7302	2866
506402	28000	350-	17	38000	1200	6002	1849
26430204	23000	300-	07	30000	1000	1002	1789
25740502	30000	400-	01	16000	1000	31002	2794
314401	35000	500-	02	23000	300	40002	1796
503102	19000	260	9046	41000	750	10002	1848
390301	21000	300-	06	19000	400	6002	1808
557234	35000	500-	01	30000	700	1002	1870
504302	42000	600	30046	10000	300	2002	1848
285401	35000	500-	0205	30000	1500-	2	2793
11310308	35000	500-	4301	23000	1500-	2	1690
573407	30000	460	114004	36000	1000	30002	1877
397202	26000	400-	14	14000	1350	502	2809
207101	60000	1000-	0802	12000	500	115002 1	1746
547205	24000	400-	02	20000	1330	1702	1866
16223003	24000	400	00601	22000	1500	3002	1710
303106	35000	600-	01	25000	2500-	2	2795
50830204	30000	550	15017	20000	1500	2002	1849
458201	17000	300-	06	20000	2000-	2	1813
507102	16000	310	4046	24000	300	21002	1849
308101	27000	500-	02	10300	300	3002	1796
313301	20000	400-	02	16000	600	20002	1796
228101	30000	600-	4953	18000	700	8002	1758
47540204	50000	1000-	01	20000	1000	20002	2814
537305	20000	400-	11	11000	700	25002	1864
281106	15000	300-	01	16000	1200-	2	1793
309102	30000	600-	01	24000	2300	7002	2796
308201	24000	500-	02	14000	1000	15002	1796

Table B-1. Listing of Bodart Battle Data, Category 21
(Continued)

315202	36000	800-	01	34000	1200-	2		2796
110230	18000	400-	07	32000	3000-	2		1689
440104	36000	800	70001	20000	2000	15002		2812
310201	30000	700-	02	35000	1000-	2		1796
51610701	22000	500-	04	18000	1500-	2		1855
50920204	34000	800	10017	40000	1200	3002		1849
508117	25000	600-	02	15000	800-	2		2849
309301	25000	600-	3452	9000	550	8502		1796
273102	38000	900-	01	22000	2000-	2		2793
290201	20000	500-	0205	30000	900-	2		1794
279305	8000	200-	01	34000	1200	19002		2793
520101	40500	1000-	02	8500	360	11402		2859
15820106	16000	400-	0815	20000	1700	23002		1709
356101	40000	1000-	02	10000	1000	30002		1800
472101	20000	500-	0504	30000	3000-	2		2814
505302	19000	500-	46	16000	700-	2		1848
23710252	38000	1000-	05	13500	1300	122002	4	2759
239101	30000	800-	3403	16000	650	2502		1760
545205	27000	720-	02	22000	1100	27002		1866
165307	260000	7000-	04	40000	2100	9002		2711
312402	44000	1200	30001	30000	2000	10002		2796
220205	22000	600-	01	41000	3400	50002		2757
268104	18000	500-	07	15000	1500-	2		1791
248101	24000	700	30004	16000	500	19002		1762
515104	21000	600-	0701	29000	1200-	2		2854
504202	31000	900-	12	16000	1400-	2	2	1848
297201	35000	1000-	02	18000	1500	10002		1794
400202	27000	800	2000126	18000	1500	10002		2809
349102	16000	500-	01	17000	700	8002		2799
283405	26000	800	10001	32000	2400	7002		2793
311301	35000	1100-	02	25000	2000	10002		1796
45711506	32000	1000	170001	38000	3600-	2		2813
208301	25000	300-	0208	41000	300	38002	1	1746
534305	37000	1200-	11	23000	1200	36002	2	1864
551205	25000	820	3026	22000	750	2002		1866
559234	30000	1000-	01	20000	700	18002		1870
253204	15000	500-	07	40000	2000-	2		1771
428301	18000	600-	06	22000	1100	5002		1811
593213	36000	1200	004	38000	3000	8002		1904
499219	15000	500-	07	20000	2000	30002		1832
23520252	26000	900	50005	13500	500	3002		2759
363201	20000	700/	02	15000	800	10002		1805
315102	28000	1000-	01	32000	1000	18002		2796
29130208	28000	1000-	01	42000	2000-	2		2794
378101	27000	1000-	04	22000	600	2002		1806
316102	26000	950	35001	24000	1000	8002		2796
44320237	35000	1300	50004	27000	1500	25002		1812

Table B-1. Listing of Bodart Battle Data, Category 21
(Continued)

562134	35000	1300	30001	25000	1400	21002		1870
45130405	21400	800-	01	18000	1500	7002		2813
566334	26000	1000-	01	41000	1100	19002		1870
46011506	42000	1600-	01	32000	1100	6002		2813
19740126	40000	1500	30002	10000	650	4502		1744
51914601	26000	1000-	02	18000	1700	5002		2859
305302	36000	1400	20001	33000	3000	18002		2795
181202	20000	800	1000146	40000	1100	60002		2734
308301	50000	2000-	02	11000	400-	2		1796
312101	25000	1000	70002	12000	900	2002		1796
285201	25000	1000-	02	15000	1500	5002		2793
354401	17000	700	30002	30000	800	11002		2800
448106	17000	700-	01	15000	1000	4002		1813
530210	26000	1100-	16	5000	200	48002	4	1863
524110	45000	1900	50016	30000	1300	3002		1862
567234	23000	1000	10001	37000	1350	11502		1871
318201	28000	1200	80002	16000	1300	87002		1797
321401	45000	2000-	02	30000	1000	40002		1797
270301	45000	2000-	02	13200	1000	5002		1792
363302	25000	1100-	01	6000	600	9002		2805
577304	22000	1000-	07	32000	1400-	2		2877
493304	11000	500-	07	20000	1500	5002		2828
34240108	22000	1000	200004	23000	2100	20002		1799
45410504	32000	1500	3000137	18000	1700	15002		2813
357301	20000	1000-	02	40000	500-	2		1800
054101	10000	500-	06	29000	800	2002		1630
48920226	20000	1000-	01	17000	500	12002		2815
292201	20000	1000-	0208	20000	600	4002		1794
48830226	40000	2000-	01	20000	1000-	2		2815
34520108	24000	1200	20004	21000	1600	10002		2799
230101	36000	1800-	3403	27000	2500	3002		1759
326101	10000	500-	48	26000	2500-	2		1798
288201	20000	1000-	06	20000	2000	15002		2794
391101	21000	1100-	06	23000	900	45002		1808
418107	30000	1600-	04	20000	1800-	2		2810
292301	24000	1300-	0253	19000	900-	2		1794
46940204	25000	1400-	01	13000	500	2002		2814
439205	16000	900	35004	22000	1500	25002		1812
19830646	24000	1400	210002	16000	700	8002		1744
427201	17000	1000-	06	23000	1000	47002		1811
249205	27000	1600-	0252	31000	3000	44002		1762
274201	20000	1200-	0246	15000	700-	2		2793
393203	15000	900	10001	20000	1500	1002		2809
429201	33000	2000-	06	33000	2000	194002	1	1811
575204	18000	1100-	07	19000	1300-	2		2877
547105	26000	1600-	0237	44000	2900	26002		1866
478302	30000	1900	100001	20000	800	12002		1814

Table B-1. Listing of Bodart Battle Data, Category 21
(Continued)

522116	32000	2000-	10	35000	1600	13002	2861
595113	20000	1250-	04	24000	2000	2002	1904
485202	11000	700	20048	29000	1700	24002	1815
525216	20000	1300-	10	18000	1800	7002	2862
413307	30000	2000-	04	15000	1000-	2	2809
402101	45000	3000-	02	25000	1900	17002	1809
082301	15000	1000-	0639	25000	2000	50002	2653
473201	30000	2000-	0402	15000	1400	36002	2814
498114	22000	1500-	04	24000	1700	23002	1831
08330639	22000	1500-	01	25000	2000	40002	1656
240134	19000	1300-	01	17000	1500	22002	1760
177306	29000	2000-	30	21000	800	23002	1719
247405	16000	1100	30002	32000	1200	7002	2762
247105	23000	1600-	02	20000	1750	12502	2762
387301	15000	1100-	06	22000	1000	30002	1808
418304	19000	1400-	07	30000	3000-	2	1810
353301	25000	2000-	02	20000	1250	27502	1800
512211	39000	3200	60034	26000	1750	11502	2850
521460	18000	1500	120048	38000	1700	30002	2860
069101	12000	1000-	06	25000	2000-	2	2642
470101	36000	3000-	0204	30000	3000-	2	1814
471301	25000	2100-	0405	39000	3000	10002	2814
215105	30000	2600	80002	33000	2200	8002	1756
47110112	34000	3000	50002	32000	2800	12002	2814
496104	28000	2500-	07	40000	3000	20002	1829
500119	22000	2000-	07	44000	3000	90002	1832
24115202	22000	2000	120005	12000	1000	1002	2760
126209	12000	1100-	3714	22000	2000	19002	1702
509111	20000	1870	3034	14000	1350	16502	2849
441101	2000	200	45015	32000	550	15502 1	1812

Table B-1. Listing of Bodart Battle Data, Category 22

272102	36000	50-	01	9000	2000	8002		2793
194302	32000	100-	26	9000	1000	9002		1743
105430	32000	100-	07	18000	3000-	2		1684
430115	40000	250	110001	1800	800	10002	1	2812
235305	30000	200-	02	4000	1800-	2		2759
390401	24500	200-	06	11000	2500	9002		1808
296401	47000	500-	03	7000	2000-	2		1794
572104	50000	600-	07	10000	1500	15002	2	1877
14610106	41000	500-	3005	19000	3000	10002		1706
419101	60000	700-	15	6000	1000	50002	1	1810
43121506	50000	600-	01	700	200	5002	1	2812
254203	25000	320-	10	11000	2000-	2		1776
336301	25000	400	20002	6000	750	16502		1799
299101	60000	1000-	06	13000	4000-	2		1794
434104	33000	600-	37	2600	300	23002		2812
29030502	50000	900-	01	5000	1000	21002		2794
478201	30000	800-	0204	13000	2300	27002		2814
305202	27000	650-	01	12000	1500	5002		2795
111130	17000	400-	07	40000	10000-	2		1689
558134	40000	1000-	01	23000	2500	170002	1	2870
307301	24000	600-	46	12000	1600-	2		1796
394201	16000	400-	15	23000	4000	4002		1809
109130	53000	1300-	07	8300	7000	13002	1	1688
15110437	36000	1000-	09	12000	1400	26002		2706
27413801	35000	1000-	0136	14000	4000-	2		2793
055209	33000	1000-	30	27000	3000-	2		2632
552134	51000	1550	5001	6000	1100	10002		1870
388106	32000	1000	100001	22000	3000	10002		2808
330302	22300	700	10001	9500	2000	6002		2799
592213	36000	1100	004	6000	1300	7002		1904
186402	40000	1300-	07	17000	2000-	2		1738
413206	21000	700-	01	11000	1300-	2		2809
594213	24000	850-	04	12000	1550-	2		1904
335201	23000	800-	02	10000	2200	30002		2799
310302	22000	800-	01	10000	1200	16002		2796
499319	14000	500-	07	17000	2500	15002		1832
05133006	28000	1000-	34	22000	4000	70002		1623
078336	28000	1000-	38	16000	3000	60002		1651
431115	51000	1850	665001	5000	1300	37002	1	2812
169101	28000	1000-	0830	2200	700	15002	1	2712
10523014	28000	1000-	07	16000	9000-	2		1683
577104	26000	1000-	07	15000	3000-	2		1877
052330	26000	1000-	1134	11000	3000-	2		1626
389315	19000	750-	01	13000	1500	3002		2808
205301	50000	2000-	0508	4000	500	54002	1	1745
288401	25000	1000-	0205	10000	1500-	2		1794
068109	26000	800-	3037	18000	3000	12002		2642
485201	24000	1000-	02	18000	2500	9002		1815

Table B-1. Listing of Bodart Battle Data, Category 22 (Continued)

13710106	26000	1100-	4430	7000	900	61002	1	1704
570301	40000	1700	50034	6300	700	1002		2871
071130	22000	1000-	0109	18000	3000	40002		1643
530101	22400	1000-	02	8800	2000	15002		1799
144230	13000	600-	17	24000	6000-	2		2705
219202	32000	1500	10005	14000	1500	4002		1757
548205	30000	1400-	02	23000	3330	22702		1856
544205	24000	1200-	02	31000	3700	21002		1666
277236	10000	500-	54	35000	5000-	2		2793
289101	30000	1500-	0302	10000	1500-	2		1794
119201	26000	1300-	06	20000	3500	22002		1594
279201	30000	1500-	08	13000	3100-	2		1793
433101	17000	900	30004	16000	1900	6002		1812
27710308	24000	1300-	01	8000	1000	70002	1	1795
267301	28000	1500-	0308	13000	2000-	2		1794
43510237	40000	2200-	04	18000	5000-	2		1812
05633000	35000	2000-	0937	25000	6000	60002		1834
395101	17500	1000-	06	23000	8000	20002		1809
22433403	31000	1800	30001	47000	5200	7002		2758
29210504	26000	1500-	14	26000	3000-	2		2794
19520302	35000	2050	45001	26000	2800	12002		2743
574407	29000	1700-	04	15000	2000-	2		1877
05620934	17000	1000-	30	15000	7000	30002		2635
381201	20000	1200/	04	16000	2500/	2		1807
34510304	40000	1800	2000108	20000	3000-	2		1799
284301	32000	2000-	03	16000	4000-	2	1	2793
097230	17000	1100-	01	14000	2500	24002		2675
13312601	23000	1500-	3005	18000	4500-	2		2703
128130	46000	3000-	01	4400	1700-	2	1	2702
339202	32000	2100-	01	11000	3100-	2	1	1799
376101	30000	2000-	05	15000	2000	40002		1806
442104	27000	1800-	01	25000	4000	30002		2612
074101	15000	1000-	06	18000	3000-	2		1645
389101	30000	2000-	0402	7000	1200	18002		1805
574304	24000	1600-	07	8000	2400	3002		2877
282354	31000	2100-	36	25000	4000-	2		1793
167201	30000	2100-	0330	18000	2500	41002		82712
27630502	43000	3000-	01	22000	4000	130002	1	2793
407201	11300	600-	06	20000	4000	5002		1809
089115	14000	1000-	06	18000	4000	60002		2665
47920204	28000	2000-	01	21000	5000	60002		1814
16233003	22000	1800-	0601	20000	5000	50002		1710
547102	29000	2150	25001	15000	3400	42002		2799
527110	24000	1800	10018	15000	1600	5002		2862
317202	20000	1500	50001	23000	3000-	2		2796
314201	20000	1500-	02	14000	2500-	2		1796
456101	40000	3000-	04	20000	6000-	2		1813
071336	20000	1500-	38	18000	6000-	2		1644

Table B-1. Listing of Bodart Battle Data, Category 22
(Continued)

531110	30000	2300	20016	18000	3000	30002		1863
05113006	26000	2000-	34	21000	8000-	2		1622
488101	32000	2500-	05	24000	2500-	2		1815
580124	32000	2500	23	30000	5000-	2		2866
238302	38700	3000-	05	10800	6000	32002		1760
284236	25000	2000-	34	25000	15000-	2		2793
496204	37000	3000-	07	15000	6000	90002	1	1629
05243006	24000	2000-	11	30000	4000	50002		1626
20910246	30000	2500	5000106	25000	4500	15002		2746
110330	60000	5000-	01	6000	2200	58002	1	2689
577204	36000	3000-	07	18000	5000	100002	2	1677
571234	26000	2200-	01	18000	4800-	2	1	1670
528110	25000	2200	40016	22000	2700	33002		1862
059101	34000	3000-	06	16000	5000	10002		2655
422301	17000	1500-	06	21000	7000	140002	1	1811
15610830	11000	1000-	01	22000	2500	5002		2703
419204	22000	2000-	07	35000	5000	50002		1810
317302	40000	3800	100001	20000	4000-	2	1	2797
15120106	21000	2000-	0308	16000	5000	70002		2707
16310830	31000	3000-	01	4000	1800	22002	1	1710
330202	16400	1600	110001	14500	1500	3002		2799
523110	27000	2660	24016	17000	2000	130002	4	1862
318101	22000	2200	100002	28000	4000	80002		1797
174202	45000	4500-	07	18000	3000	120002	1	1715
306301	25000	2500	5000246	18000	3000	40002		1795
148208	30000	3000-	0106	5500	1000	40002	1	1706
311101	20000	2000-	02	15000	3000-	2		1796
275354	20000	2000-	36	20000	5000-	2		1793
265354	20000	2000-	36	25000	8000-	2		2793
16610830	30000	3000-	01	5000	1800	32002	1	1711
443104	28000	3000-	01	25000	2200	8002		2812
281401	45000	5000-	0308	30000	2500	5002		2793
506102	28000	350-	46	42000	700-	2		1648
482101	14000	1600	4001506	25000	1700	3002	1	2814
407101	35000	4000-	02	37000	3500	65002		1809
593113	40000	4600-	04	15000	900-	2		1904
459405	16000	1900	20001	15000	900	10002		2813
536216	25000	3000-	10	30000	3000	15002		2864
28920302	22500	2800-	01	44000	4000-	2		2794
278101	24000	3000-	0308	16000	1600	14002		2793
377401	26000	3300	70004	44000	3500-	2		1806
545102	27000	3600	120005	32000	1300	1002		2866
242301	20000	2700	40034	16000	1300	5002		2759
149130	30000	4300-	0106	42000	3800	32002		2706
540116	37000	5600	70010	27000	1300	14002		2864
401301	30000	4600	140002	30000	2400	22002		1809

Table B-1. Listing of Bodart Battle Data, Category 22
(Continued)

316201	20000	3500	130002	24000	2200	40002	1796
154209	7000	1600-	04	33000	900	6002	1706
494204	4500	1500-	07	35000	2500	5002	1328

Table B-1. Listing of Bodart Battle Data, Category 23

539110	43000	4400	60016	13000	2000	20002		1864
539210	40000	4100	160016	18000	1900	12002		1864
06513006	14000	1500-	01	20000	6000	30002		1639
127230	25000	2700-	0108	30000	3500-	2		2702
240305	30000	3300	50002	30000	3800	22002		2760
437301	36000	4000-	04	20000	6000-	2		1812
06230901	14000	1600-	30	17000	2000	14002		2638
095301	22000	2500-	30	32000	4000-	2		1674
411301	21000	2400-	06	23000	3300	20002		1809
367102	49000	5700-	01	46000	6300	17002		2805
498204	38000	4500	50014	52000	7000	20002		2631
343101	33500	4000-	04	23000	6000	20002		1799
433201	28000	3400	30004	20000	2700	11002		1812
13130108	19000	2300-	08	15000	2500	8002		1703
073109	16000	2000-	30	16000	4000	45002		2645
395201	16000	2000-	15	30000	8000-	2		1809
353134	35000	4500	40001	28000	4100-	2		1870
381301	45000	8000-	0504	18000	3000-	2	1	1807
16210330	60000	8000-	01	8000	3000	50002	1	1710
11430308	20000	2700-	0145	25000	4400-	2		1691
53720204	37000	5000	50001	33000	9500	70002		2799
16010308	40000	5400-	01	7000	5200	7002	1	1709
143201	22000	3000-	30	24000	4000	5002		1705
097130	22000	3000-	01	14000	4000-	2		2675
05510937	58000	5000-	3006	34000	8000	40002		2631
101101	30000	4400-	0808	30000	7000	40002		2677
206205	35000	5100-	3702	35000	3800	67002		1745
1821014E	40000	5900	10002	27000	5400	2002		1734
192205	28000	4200	80002	28000	3000	53002		1742
116201	46000	7000-	06	8300	4000	43002	1	1692
19920106	26000	4000-	0248	25000	3600	9002		1744
128301	17000	2700-	30	14000	1850	10502		1702
486103	32000	5100	50001	21000	4400-	2		2815
454104	10000	1600-	07	30000	5000-	2	1	2828
282236	25000	4000-	54	40000	8000-	2		2793
36830402	25000	4000-	01	8000	5000	16002		2305
275236	12000	2000-	34	58000	5000-	2		2793
16712008	18000	5000-	01	5500	2000	35002	1	1712
205105	22000	3700	3000237	38000	4500	30002		2745
070101	23000	4000-	06	25000	8000	60002		2643
573307	20000	3500-	04	33000	7500-	2		1877
43530126	34000	6000-	04	22000	6000-	2		1312
189205	21600	3900	70002	15800	3000	15002		2741
300104	22000	4000-	14	28000	8000	120002	2	2794
262203	7000	1300-	0106	31000	6000-	2	1	2782
212301	35000	6500-	08	26000	5500	205002	1	1747
434204	23000	4300-	01	20000	3700-	2		2812

Table B-1. Listing of Bodart Battle Data, Category 23 (Continued)

280506	17000	2000-	01	22000	4000	15002		1793
164201	21000	4000-	3008	13600	3000	20002		2710
114107	50000	9500-	30	5000	4500-	2	2	1690
134201	26000	5000-	30	5600	1800-	2	1	1703
315302	28000	5600-	01	21000	3000-	2		2796
05610937	20000	4000-	30	25000	5000-	2		2632
069309	25000	5000-	30	30000	10000	50002		2642
42511506	32000	7000	50001	18000	8000-	2		2811
134101	18000	4000-	3008	22000	4000	20002		1703
123309	9000	2000-	0414	40000	8000	120002		1700
268204	18000	4000-	07	15000	8000	70002	2	2791
355301	28000	6500	150002	31000	7000	40002		1800
518204	30000	7000-	07	16000	3000-	2	1	2855
51520301	30000	7000-	04	36000	12000-	2		1854
425201	18000	4300-	06	18000	8000	100002	1	1811
13630308	25000	6000-	2601	11000	2000	20002		1704
05033008	20000	5000-	34	20000	5000	8002		1621
170301	40000	10000-	30	7000	2000	50002	1	1713
441201	24000	6000-	04	24000	8000-	2		1812
16330830	28000	7000-	01	7000	3400	36002	1	1710
436201	35000	8800-	04	25000	6000-	2		1812
058230	30000	8000-	09	4000	1400	12002	1	1634
077101	14000	4000-	0620	18000	5000	50002		2648
14924430	10500	3000	20000106	36000	14000-	2	1	2706
574104	18000	9400-	07	27000	7000-	2		2877
495104	20000	6000-	07	15000	8000	70002	1	1828
440204	32000	10000	200001	27000	6000	20002		2812
267304	32000	10000-	07	40000	31000	90002	2	2790
061209	22000	7000-	3037	30000	10000	80002		2636
07420134	18000	6000	150030	16000	4000	20002		2645
122101	30000	10000-	06	12000	7000	50002	1	1697
072101	20000	8000-	30	16000	4000-	2		2644
15813003	35000	14000-	01	16000	7000	90002	1	1708
415301	34000	15000-	06	9400	5200	42002	1	1309
420204	25000	12000-	07	15000	6000-	2	1	1810
171306	40000	20000-	45	16000	6000-	2	1	1714

Table B-1. Listing of Bodart Battle Data, Category 31

235405	12000	50-	02	9000	100	14002		2759
505146	23000	100-	02	6000	300	13002		2848
333402	19000	100-	01	8000	300	31002		2799
587363	5500	30	003	13500	910	1502		2899
246105	17000	100-	02	3000	200	15002		1762
256202	3200	20-	05	10000	800-	2		2779
289402	9000	50-	01	6000	600	6002		2794
124130	15000	100-	01	5000	350	1502		1701
394301	12700	100-	06	12500	500	25002		1809
502111	11000	100-	34	6000	150	8002		1848
189105	5000	50-	02	1200	60	10402	2	2741
510204	4000	50-	17	4000	200	12002		1849
393101	15500	200-	06	11300	1000	60002		1309
33410402	11000	150-	01	8000	500-	2		2799
549205	14000	200-	0249	7000	510	24902		1866
206105	8500	120-	37	5500	500	9002		2745
349402	13000	200-	01	4000	200	5002		2799
431305	6600	100-	04	6000	300	3002		1812
504146	18000	500	5002	11000	400	2502		2848
416301	6000	100-	06	7500	500	57002		1310
531316	18000	300-	10	9000	500	40002		1863
237202	12000	200-	05	4000	300	17002		2759
371201	12000	200-	0537	9000	700	11002		1806
509317	15000	250-	02	8000	800-	2		2849
168101	22000	400-	0805	7000	200	23002	1	2712
195102	5400	100-	01	6000	600	1502		2743
317101	15000	300-	02	14000	400	18002		1796
502346	25000	500-	02	5000	200	6502		2848
46930405	15000	300	001	4000	400	6002		2814
072309	16000	400-	30	12000	1000	16002		1644
586303	8500	200-	63	2500	40-	2		1899
191202	4000	100-	26	7000	150	5002		2742
353202	8000	200-	01	5000	200	15002		2800
146108	20000	500-	0106	5000	300	47002	1	1706
342101	8000	200-	02	5000	300	7002		1799
435201	20000	500-	04	7000	700	3002		1812
587165	1500	40	005	3500	85	6452		2899
330402	15000	400-	01	7000	400	11002		2799
348201	15000	400-	02	15000	1200-	2		1799
475401	15000	400-	0204	10000	800	12002		2814
247234	16000	450	1503701	13000	600	16002		2762
412404	16000	500-	07	12000	1000	7002		1809
392301	14000	400-	05	15000	1000	12002		1808
416201	27000	800-	06	2800	200	28002	1	1810
587263	4000	120	2005	11500	900	1002		2899
227301	10000	300-	03	8000	800	22002		2758
193101	18000	550-	02	6500	250-	2		1742

Table B-1. Listing of Bodart Battle Data, Category 31
(Continued)

502234	16000	500-	11	11000	800-	2		2348
429106	4500	150-	01	10400	600-	2	1	2811
346201	12000	400	80002	5000	300	10002		1799
304502	6000	200-	01	12000	1000	5002		2795
405206	9000	300-	01	8000	700-	2		2809
064309	13000	600-	30	10000	1000	4002		2639
209301	20000	700-	08	9000	400	36002	1	1746
348102	14000	500-	01	9000	500-	2		2799
347201	11000	400-	02	12000	1000-	2		1795
563201	14000	500-	34	8000	800	5002		2370
550202	3000	110-	12	9200	700	11002		2866
391301	8000	300-	06	12000	200	20002		1808
586203	8700	350-	63	1500	90-	2		1899
152401	24000	900-	30	3500	300-	2	1	1703
303301	13000	500-	0336	17000	1700	63002		2795
513101	20000	750	002	10000	1000	20002		1796
234105	5000	200-	5202	10000	350	8502		2759
350101	5000	200-	02	7000	250	14002		1800
283236	5000	200-	54	25000	1500-	2		2793
469105	12000	500-	01	10000	300	5002		2814
285502	7000	300-	01	15000	1200-	2		1794
215305	15000	700-	02	14000	400	5002		1757
586403	10200	500-	63	6000	20-	2		1899
374101	16000	600-	05	13700	1000	42002		1806
332301	8000	400-	02	8000	600	20002		1799
329201	6000	300-	02	6000	500	14502		1799
333301	12000	600-	04	10000	1000	7002		1799
367201	12000	600-	02	4000	400	18002		1805
307401	17500	900-	02	9500	400	17002		1796
585103	4500	230	22065	1500	110	02		1899
357102	37000	2000	110001	35000	1200	10002		2800
573107	11000	600-	04	16000	1000-	2		2377
262303	20000	1100-	01	10000	700-	2		1782
526216	8000	450-	10	16000	1050	43502		2862
298101	18000	1000-	0246	11000	800-	2		1794
558234	9000	500-	01	20000	2000	1002		1870
549105	16000	950	5026	14000	750	6502		1866
512311	4000	250-	34	12000	300-	2		2850
278101	12000	800-	06	15000	200-	2		2793
226434	3000	200-	01	6000	400	4002		2758
369201	9000	600-	02	4800	400	44002	4	2805
363101	7000	500-	02	8000	400	16002		1805
140330	7000	500-	14	20000	2000-	2		2704
254303	6000	430-	10	2000	200	13002	2	1776
358201	11000	800	20002	10000	700	6002		1800
096101	4000	300-	30	5000	300	9002		2674
424201	4000	300-	06	11000	900	15002		1811

Table B-1. Listing of Bodart Battle Data, Category 31
(Continued)

585203	3400	200	1063	1200	90	2002	1899
295201	10000	800-	06	20000	1400-	2	1794
409101	5000	400	50002	22000	2000-	2	1809
422101	6000	500-	06	12000	1000	52002	1811
558334	6000	500-	01	10000	900	6002	1870
371101	12000	1000-	4803	11000	1000-	2	1 2806
352201	6000	500-	02	4000	400	7502	1800
534102	5000	450-	11	5000	420-	2	2864
274301	10500	900-	02	6000	600-	2	1793
243201	12000	1100-	34	10000	700	6002	1762
544153	16000	1500-	05	9000	900	9002	1866
518301	8300	300	10002	18700	1100	4002	2859
484303	5000	500-	10	7000	300-	2	1814
241409	3000	300	50005	5000	300-	2	1760
344401	3200	320	83004	7000	450	17002	2799
523210	12000	1200	20016	16000	1100	5002	1862
588363	5000	500-	03	12000	1110	1102	2900

Table B-1. Listing of Bodart Battle Data, Category 32

485110	6000	20-	03	12000	2000-	2	2815
254410	2400	10-	03	1500	1300-	2	2776
110130	20000	100-	07	5000	2000-	2	1689
506202	16000	100-	17	6000	1000	12002	1848
511204	11000	100-	17	8000	1100	12002	1849
273205	10000	100-	01	1200	300	9002	1793
510304	20000	200-	17	8000	2000-	2	1849
327301	9600	100-	02	3400	3000-	2	1799
319201	9000	100-	57	7000	800	12002	1797
442304	4500	50-	14	3500	500	8002	2812
412125	18000	200-	0126	10000	1700-	2	2809
590163	1500	20	063	1200	160	4402	2900
346302	15000	200-	01	7000	1000-	2	2799
284106	6000	100-	01	10000	2500-	2	1793
447301	8000	100-	06	4000	1700-	2	1813
142308	14000	200-	01	15000	2000-	2	1705
345301	7000	100-	02	3000	1200-	2	2799
266202	7000	100-	07	7000	2000-	2	1790
27120549	13000	200-	01	1500	180	15202 3	2792
239234	13000	200-	37	8000	1200	27002	1760
408201	13000	200-	06	12000	2000-	2	1809
324101	20000	300-	19	6000	2000-	2	1798
451205	15400	250-	01	5000	1400-	2	2813
580413	6000	100-	21	5000	500-	2	2894
510104	12000	200-	17	7000	1300	5002	2849
417201	6000	100-	06	1800	400	14002 1	1810
579204	18000	300-	07	10000	2400-	2	1878
406402	11000	200-	14	6300	1100	6002	1809
408202	15000	300-	26	5000	600	2002	1809
572204	10000	200-	07	7000	1000-	2	2877
510417	10000	200-	02	6000	1000-	2	2849
318302	15000	300	30001	6000	1100	13002	2796
054209	20000	400/	30	8000	1700	8002 2	1631
057309	5000	100-	30	6000	1500	3002	2634
342202	20000	400-	01	7500	2000-	2	2799
168301	25000	500-	0830	3200	1000	22002 1	2712
428103	5000	100-	01	3000	2000-	2	2811
50110703	4700	100-	19	5000	2000	12002 2	1840
329101	4500	100-	02	6500	1000	40002	1799
327401	9000	200-	02	6000	1100-	2	1799
430201	8000	200-	06	10000	1200-	2	1812
327201	12000	300-	02	2200	400	11002	1799
280201	8000	200-	06	6000	1200-	2	2793
282402	8000	200-	01	4000	1200	18002	1793
277454	20000	500-	36	8000	4000-	2	2793
285336	20000	500-	54	7000	6000-	2	2793
144330	2000	50-	26	4000	3500-	2	1705
256302	11000	300-	05	3000	400	11002	1779

Table B-1. Listing of Bodart Battle Data, Category 32
(Continued)

280454	11000	300-	36	6000	3000-	2		2793
172104	18000	500-	09	5000	2500-	2		1715
224202	3500	100-	05	2800	750	2502		2756
258203	10000	300-	10	7000	1000	60002	1	2780
244305	10000	300-	04	4000	600	16002		2761
207303	10000	300-	50	6000	1000	5002		1746
328202	6500	200-	01	7000	1000-	2		2799
368101	13000	400-	02	3000	1500-	2		1805
243305	12500	400-	3701	5500	600	20002		2761
255103	18000	600-	10	8000	900	4002		1777
255303	15000	500-	10	10000	1200-	2		1777
389215	15000	500-	01	4400	600-	2		2808
47520112	12000	400-	02	4000	600	17002		2314
245105	12000	400-	04	15000	2500	4002		2761
473101	18000	600-	0402	9000	2400-	2		2814
139430	3000	100-	17	12000	4000-	2		1704
503202	19000	650	5046	4000	700	32002		1848
338202	11000	400-	01	3100	400	27002	1	1799
389303	12000	430	2063	2000	400	02		1900
287401	14000	500-	53	2000	500	3002	1	1794
237303	5400	200-	01	2800	500	1002		2760
480301	13000	500-	37	9000	1000-	2		2814
285106	8000	300-	01	5000	4000-	2		1793
392201	15000	600-	06	9000	1000	15002		1808
411201	10000	400-	06	8000	1000	6002		1809
350401	5000	200-	02	3000	400	14002		1800
205245	2500	100-	03	2300	400	10002		2745
250404	10000	400-	07	16000	3000	4002		1770
278302	2500	100-	01	8000	2000	20002		1793
146301	5000	200-	30	2000	600-	2	1	1706
392101	25000	1000-	06	3500	1500	20002	1	1808
135417	10000	400-	30	2400	1600-	2		2704
130226	12000	500-	30	10000	1200	13002		1703
28830302	12000	500-	01	10000	3000	5002		2794
340101	12000	500-	02	6000	3000-	2		1799
157308	7000	300-	2601	15000	2000	8002		2708
246402	7000	300-	05	5000	1400-	2		1762
412203	15000	650-	01	3000	1000	20002	1	1809
21214602	7000	300-	01	14000	5700	5002		2747
388301	7000	300-	1506	3000	2000-	2		1808
249334	17000	750	35001	6000	700	53002	1	2762
207406	9000	400-	02	4000	1600-	2		1746
335402	4300	200-	01	3300	400	6002		2799
446304	15000	700-	01	12000	2000-	2		2813
377301	18000	850-	04	5500	1400-	2		1806
201102	17000	800-	0126	7000	2400-	2		1745
179406	16600	800-	30	6200	1000-	2		1734

Table B-1. Listing of Bodart Battle Data, Category 32 (Continued)

131109	2000	100-	3714	12000	1300	7002		1703
294301	12000	600-	06	12000	1300-	2		1794
415101	12000	600-	06	18000	2000	20002		1809
327104	8000	400-	01	4000	500	5002	1	2798
423315	10000	500-	01	12000	1500-	2		2811
272301	10000	500-	02	6000	800-	2		2793
28710308	2000	100-	01	7000	1300-	2		2794
127101	4000	200-	01	2800	600	4002		2702
467111	10000	500-	0409	4000	1100-	2		2813
350302	20000	1000-	01	7000	2000-	2		2800
340201	12000	600-	02	8000	2400-	2		1799
432101	4000	200-	06	10000	3000-	2		1812
155130	10000	500-	17	16000	6000	5002		2708
437104	6000	300-	05	1500	800-	2		2812
427301	19500	1000-	06	3000	400	26002	1	1811
448405	4800	250-	01	2600	300	4502		2813
343201	11500	600-	0264	10000	1500	35002		1799
067401	7500	400	30	9000	4000	30002		2542
225102	11000	600-	05	7000	900	15002		2758
226203	11000	600-	01	6700	1100	56002	1	1758
449204	20000	1100-	01	8500	2100	8002		2813
17220537	22000	1200-	09	7000	2000-	2		1715
136230	3600	200-	17	7400	800-	20002		1704
477302	8000	450	45001	5000	800	5002		1814
442204	3500	200-	01	2000	250	17502		2812
280354	7000	400-	36	18000	5000-	2		2793
059309	7000	400-	37	7000	2400	26002		1635
306401	14000	800-	32	4000	2500-	2		1796
326201	3500	200-	48	4500	4000-	2		1796
379101	12000	700	40004	9000	1100	3002		1807
581113	12000	700	021	15000	3500	50002		2894
574204	17000	1000-	07	11000	1400-	2		2877
075306	17000	1000-	01	12000	4000-	2		2646
466305	10000	600-	01	4000	500	10002	2	1813
480101	10000	600-	04	8800	1500-	2		2814
350202	10000	600-	01	3500	1000-	2		2800
485401	10000	600-	05	5000	1600-	2		1815
45830405	12000	1100-	01	4500	600	19002		2813
566134	16000	1000-	01	11000	1000	7002		1870
341204	8000	500-	0108	11000	1500-	2		2799
280101	8000	500-	46	6000	1000-	2		2793
341101	8000	500-	02	4400	2200-	2		1799
364201	16000	1000-	02	8000	4000-	2		1805
578107	20000	1300-	04	5000	1900-	2		2877
271401	12000	800-	46	7000	800-	2		2793
335302	15000	1000-	01	7000	800-	2		1799
150101	15000	1000-	30	10000	1500	25002		1706

Table B-1. Listing of Bodart Battle Data, Category 32
(Continued)

157201	3000	200-	05	1200	200	10002		2708
411125	1500	100-	52	2300	500	7002		2809
575304	12000	800-	07	5000	1500-	2		2877
058137	15000	1000-	30	12000	4000	4002		2634
108330	3000	200-	07	15000	5000	20002		1686
159301	6000	400-	30	7000	2600-	2		1709
073230	10000	700-	0105	11000	1500	20002		2645
417104	23000	1600-	07	5000	3000	20002	2	1810
528310	14000	1000	25016	11000	1000	3502		2862
244101	14000	1000-	54	7000	300	12002		1781
338101	14000	1000-	02	8000	1000	13002		1799
472304	7000	500-	01	4000	500	35002	2	1814
414204	7000	500-	07	9000	1500	3002		1809
351402	14000	1000-	01	7000	2000-	2		2800
370102	11000	800-	26	7000	1000-	2		1805
37040348	4800	350-	01	4300	1700-	2		2808
351101	4000	300-	02	4500	500	15002		1800
27450246	8000	600-	01	16000	2000-	2		2793
394101	13300	1000-	06	11700	1500	15002		1809
286501	20000	1500-	0246	6000	3000-	2		2794
416401	13000	1000-	06	5000	1700	73002	1	1810
534202	5500	430-	11	6000	900	1002		2864
307101	9000	700-	0246	4000	3000-	2		1795
052209	10000	800-	14	15000	2200-	2		2626
06210901	5000	400-	30	4000	700	23002		2638
494307	20000	1600-	04	6000	1400-	2		2328
084209	2500	200-	04	10000	5000-	2		1657
234303	8000	650-	01	4500	1500-	2		1759
421101	12000	1000-	06	11000	1500	95002	1	1810
276254	12000	1000-	36	14000	2000	30002		2793
353401	6000	500-	02	4000	600	14002		1800
560334	6000	500-	01	8000	1400	12002	2	1870
068206	18000	1500-	01	10000	2000	25002		1642
44710405	3000	250-	0137	2500	500	20002		2813
336101	6000	500-	02	4000	1000	10002		1799
076301	12000	1000-	06	11000	3000-	2		2648
056440	6000	500-	09	6400	2000	5002		2675
069236	12000	1000-	38	10000	4000-	2		1642
358401	6000	500-	02	3000	1450-	2		1800
263104	12000	1000-	07	5000	3000	20002	2	2787
258404	12000	1000-	14	10000	6000-	2		2794
351302	6000	500-	01	3500	1100-	2		2800
493204	7000	600-	07	11000	3500	15002	1	2828
359303	15000	1300-	01	5000	700-	2		2801
174407	23000	2000-	02	1000	600	4002	3	2717
434304	9000	800-	01	10000	1200	3002		2812
077336	11000	1000-	38	23000	3000	100002		2650

Table B-1. Listing of Bodart Battle Data, Category 32
(Continued)

078101	11000	1000	03106	9000	2000	30002		2650
472201	11000	1000-	0405	16000	5000	10002		2814
14013704	6500	600	40009	9000	2000-	2		2704
354102	7000	650-	01	5000	1000-	2		2800
126304	15000	1400-	09	4000	1200	4002		2702
061101	10000	1000-	06	12000	2000	3002		2636
07620901	20000	2000-	30	10000	1800	2002		2647
566234	4000	400	10001	8000	1800	4002		1570
172407	10000	1000-	02	3000	700-	2		2715
187302	10000	1000-	07	20000	5000-	2		1738
06730630	10000	1000-	01	11000	3000	45002		1641
125104	12000	1200-	09	5000	1400	4002		2702
28630802	10000	1000-	01	7000	2000-	2		1794
060309	5000	500-	30	3000	1000	5002		2636
16320830	9000	900-	01	3000	1000	20002	1	1710
499119	20000	2000-	07	4000	1400	26002	1	1831
252204	10000	1000-	07	10000	4000	60002	1	2771
307201	15000	1500-	02	8000	3500-	2		1796
346401	1000	100-	07	8000	4000-	2		1799
063101	5000	500-	06	7000	4000-	2		1638
214101	1000	100-	03	2300	1500-	2		1755
475102	6000	650-	01	11000	1000	3002		2814
229203	2600	300-	01	6500	400	1002		2758
12520106	10000	1200	30030	8000	800	4002		2702
255210	5000	600-	03	6000	600-	2		2777
24520204	14000	1700-	05	4000	400	36002	2	1761
352101	8000	1000-	02	10000	950	13502		1800
406101	12000	1600-	02	9000	700	15002		1809
512411	3300	450-	34	11500	700-	2	2	1850
495304	7000	1000-	07	20000	1000-	2		1828
053330	7000	1000-	11	6000	600	10002		1628
309201	14000	2200-	3452	8000	800	2		1796
501310	4400	720-	59	20000	2000-	2		1847
325301	3000	500-	19	12000	1000-	2		1798
268414	6000	1000-	04	10000	1000-	2		2792
325101	1100	200-	03	5000	400	8002		1798
247302	8000	1500-	05	10000	600-	2		2762
521160	3500	700-	48	5000	200-	2		2860
263207	8000	1600-	02	8000	550-	2		2788
229303	4000	1000	40001	7300	600-	2	1	2758

Table B-1. Listing of Bodart Battle Data, Category 33

464106	19000	2000-	01	3500	1350	21502	1	2813
315402	19500	2100	150001	10500	4400-	2		2796
321301	11000	1200-	02	8000	1000	35002		1797
066101	9000	1000-	06	18000	2000-	2		2639
332202	18000	2000-	01	8500	1000-	2		2799
355102	9000	1000-	02	17000	2000	80002		1800
225301	3600	400-	03	15000	2000-	2		2758
506317	9000	1000-	02	8000	2000-	2		2849
294101	9000	1000-	05	6000	1500-	2		1794
238201	7000	800-	03	6000	1200-	2		2760
351202	7000	800-	01	5000	1000-	2		2800
359203	5500	650-	01	2000	350-	2		2801
132201	4200	500-	30	5000	800-	2		1703
406302	5000	600	50001	18000	3000-	2		2809
328402	7500	900-	01	12000	3000-	2		2799
138104	25000	3000-	09	4500	2500	20002	1	2704
471201	5000	600-	04	4000	2400-	2		2814
169309	14000	1700-	11	12000	4000	20002		2712
27730308	9000	1100-	01	11000	2000-	2		1793
522216	10000	1230	7010	6000	1140	1602		1861
452201	4000	500-	06	7000	1000-	2		1813
16410106	12000	1500-	0315	4000	600	34002	4	2710
19820106	24000	3000-	46	6000	1100	4002		1744
331201	4000	500-	07	26000	6000	5002		1799
059201	4000	500-	0630	6000	1500	20002		2635
102309	8000	1000-	11	7000	2000-	2		2677
360103	12000	1500-	01	10000	3000	5002		2801
297304	8000	1000-	14	12000	4000-	2		2794
421304	8000	1000-	07	10000	4000	14002	2	1811
124409	6000	800-	04	8000	1000-	2		1701
551357	3000	400-	01	7000	1000	14002		2867
572307	15000	2000-	04	10000	1600-	2		2877
537216	6000	800-	10	18000	3000-	2		2864
253404	6000	800-	07	10000	2000	2002		1773
426201	15000	2000-	06	7000	2000	50002	4	1811
084401	15000	2000-	0639	12000	4000	30002		2658
450101	13400	1800-	0504	7400	1200-	2		1813
576304	25000	3500-	07	4000	1500	25002		1877
087115	14000	2000-	18	16000	4000	60002		2563
143109	7000	1000-	04	20000	5000-	2		1705
573207	14000	2000-	04	10000	3000-	2		1877
580362	11000	1600	20020	14000	3400	8002		2891
583310	12700	1850	006	1300	650	1502		1898
124209	8000	1200-	1404	20000	2800-	2		1701
13520106	10000	1500-	30	4000	600	3002		1704
183106	10000	1500-	30	6000	1000	50002	1	1734
455205	12000	1800-	01	11000	2000	60002		2813

Table B-1. Listing of Bodart Battle Data, Category 33
(Continued)

258303	2000	300-	10	4000	900	10002		1780
323203	10000	1500-	43	12000	3000-	2		1798
380101	10000	1500-	04	7000	2000-	2		1807
051234	13000	2000-	0630	17000	2000-	2		2522
130309	1300	200-	14	6700	2400	6002		2703
408102	11000	1700	30014	5000	1000-	2	4	1809
094301	9500	1500-	30	17500	2500-	2		2674
416101	5000	800-	06	12000	3000-	2		1816
177106	9500	1500	20030	6000	1500	3002		2718
102101	8000	1300-	06	12000	3000	7002		1677
214301	12000	2000-	03	2800	400	24002	1	1756
113201	12000	2000-	44	18000	2800	12002		1690
344201	3000	500-	02	9000	1500-	2		2799
422206	12000	2000	80001	10000	2500	5002		2811
145109	12000	2000-	1404	18000	5000	80002		1706
427107	6000	1000-	04	6000	2100-	2		2311
286114	6000	1000-	04	5000	2000-	2		1794
155405	12000	2000-	04	6000	3000-	2		2706
314301	7000	1200	80002	5000	2000-	2		1796
155317	7000	1200-	30	3500	3000-	2		1708
115201	4000	700-	0806	12000	1500	4002		2691
493104	17000	3000-	07	8000	1500-	2	1	1828
248305	17000	3000-	02	12500	3500	9002	1	2762
338401	6000	1100-	07	18000	12000-	2		1799
477101	2700	500	10003	9000	1900	21002		2814
132108	7000	1300-	0106	2500	700-	2		2703
484210	4000	770	15003	5000	650	2502		2814
066201	10000	2000-	06	18000	4000	20002		2640
059401	5000	1000-	3006	7000	1700	3002		2635
060101	3000	600-	06	4000	1500	5002		2635
216105	5000	1000-	02	3000	1500-	2		1757
252104	12000	2500-	07	12000	6000	60002	1	2770
152330	3500	750	006	5000	750	22502	1	2707
161309	14000	3000-	11	11000	4000	20002		2710
08543040	9000	2000-	09	6000	2000	40002		1659
389106	13000	3000-	01	15500	3500-	2	1	2308
382204	12000	2800-	01	6000	1500-	2		2807
156204	17000	4000-	09	13000	5000	10002		2708
194206	13000	3200	8000246	11000	1600	4002		1743
355201	12000	3000-	02	16000	2100	22002		1800
258403	2400	600-	10	5000	700-	2		2761
061301	12000	3000-	06	16000	4000-	2		2637
100309	16000	4000-	11	12000	4000-	2		2676
133309	14000	4000-	1457	6000	1000	50002	1	1703
34330402	21000	6000-	01	9000	2000-	2		2799
178130	18000	5200-	06	4000	900	31002	1	1719
226101	7500	2200-	4953	4500	800	5002		1758

Table B-1. Listing of Bodart Battle Data, Category 33 (Continued)

190204	10000	3000-	09	5000	2500	15002		2741
186430	2500	800-	0106	8000	2500-	2	1	1711
513107	3000	1000-	04	7000	1000-	2		2853
079101	12000	4000-	39	6000	2000-	2		1652
513304	6000	2000-	07	10000	3600-	2		1854
186204	6000	2000-	07	20000	10000-	2	1	2737
334207	5000	2000-	01	12000	4000-	2	1	2799
183206	6000	2500-	30	3000	1400	16002	1	1735
588163	450	200	003	5000	760	2902		2899
384305	6000	3000-	01	14000	5000-	2	1	2807
076106	4000	2000-	01	16000	8000-	2	1	2647
254103	2000	1150-	10	1200	450-	2		1775

APPENDIX C

COMPUTER PROGRAMS

This appendix contains the complete FORTRAN IV programs used in this study.

Computer Program for Regression Analyses

The FORTRAN program requires one Hollerith card for identification and one lead or instruction card preceding each group of data to be analyzed. An end of file card is required after each group of data cards with two end of file cards required after the final data group.

The format for the lead card is as follows:

Column
1. Code (Model) for 1st run.
2. Code (Model) for 2nd run.
3. Code (Model) for 3rd run.
4. Code (Model) for 4th run.
5. Code (Model) for 5th run.
6. Code (Model) for 6th run.
7. Code (Model) for 7th run.
8. Not used.
9. Not used.
10. Not used.
11. Printout code for 1st run.
12. Printout code for 2nd run.
13. Printout code for 3rd run.
14. Printout code for 4th run.
15. Printout code for 5th run.
16. Printout code for 6th run.
17. Printout code for 7th run.
18. Not used.
19. Not used.
20. Not used.

The code used to identify the model selected for computation is

1 = Control Model,

2 = Attacker-Defender Model,

3 = Initial Force Model.

The code used to control the form of printout desired is

0 = Summary printout only,

1 = Detailed printout.

If an end of file card is encountered followed by a Hollerith card, a lead card, and more data cards, the program first completes the computation for the data group presently in the computer and writes the results on output tape as instructed by the lead card. The memory locations used as accumulators are then reset to 0.0 and the computer is instructed to read a new set of data and proceed with the computation. The program terminates when two end of file cards are read in succession. A flow diagram for the computer program is shown in Figure C-1. Tables C-1, C-2, and C-3 are sample printouts for the three models studied.

<u>FORTRAN Program Listing</u>

```
      PROGRAM REGRES

        REGRESSION ANALYSIS NO. 1

      COMMON/2/ HOL(12),IDYR(1500),M3(1500),M(1500),A(1500),R(1500),
     1ARG(1500),Y(1500),ID(1500),U(1500),DA(1500),Y(1500),ESTY(1500),
     2ESTU(1500),KRCT(1500).KACT(1500),ISSN(10),ISPN(10),Z(10),W(10),
     3ETY(10),CT(10),PU(10).PL(10),ETU(10),T(10),AU(10),AL(10)
    5 LINE=56
      REWIND 1
```

```
      SUM1=0.0
      SUM2=0.0
      SUM3=0.0
      SUM4=0.0
      SUM5=0.0
      S1=0.0
      S2=0.0
      S3=0.0
      S4=0.0
      S5=0.0
    6 KK=1
      KN=1
      READ(5,9000)HOL
 9000 FORMAT(12A6)
      READ (5,9001) (ISSN(I),I=1,7), (ISPN(I),I=1,7)
 9001 FORMAT (7I1,3X,7I1)
      PRINT 9010,(ISSN(I),I=1,7),(ISPN(I),I=1,7)
 9010 FORMAT (6H ISSN=,7I2,10X,6H ISPN=,7I2)
      I=0
      K=1
      CK2=1.0
      ISSI=ISSN(KN)
      ISI=ISPN(KN)
C   READ DATA TAPE-1ST MODEL-
      ASSIGN 98 TO KEOF
   15 READ(5,9002)II,TXO,TCX,TYO,TCY,M1,IYR
 9002 FORMAT(I4,4X,2F7.0,11X,2F7.0,11X,I1,I3)
      IF(EOF,5)100,500
  500 I=I+1
C   WRITE INTERMEDIATE TAPE-1ST TIME THRU CARDS-
      WRITE (1) I,TXO,TCX,TYO,TCY,M1,IYR
      GO TO 18
C   READ INTERMEDIATE TAPE-SUBSEQUENT MODELS-
   17 READ (1) I,TXO,TCX,TYO,TCY,M1,IYR
      IF(EOF,1)100,18
   18 GO TO (30,20,25),ISSI
   20 IF(M1-1)30,30,35
   25 IF(TXO-TYO)35,30,30
   30 XO=TXO
      YO=TYO
      CX=TCX
      CY=TCY
      M(I)=0
      GO TO 40
   35 XO=TYO
      YO=TXO
      CX=TCY
      CY=TCX
      M(I)=1
   40 IDYR(I)=IYR
      ID(I)=II
      M3(I)=M1
      A(I)=(XO-CX)/XO
```

```
      B(I)=(Y0-CY)/Y0
      ARG(I)=X0/Y0
      X(I)=LOGF(ARG(I))
      SUM2=SUM2+X(I)
      TEMP=S2+SUM2
      SUM2=(S2-TEMP)+SUM2
      S2=TEMP
      SUM3=SUM3+X(I)**2
      TEMP=S3+SUM3
      SUM3=(S3-TEMP)+SUM3
      S3=TEMP
      N=I
      L=I
   45 DO 55 J=N,L
      U(J)=(1.0-A(J)**CK2)/(1.0-B(J)**CK2)
      DA(J)=U(J)*ARG(J)**CK2
      Y(J)=LOGF(DA(J))
      SUM1=SUM1+X(J)*Y(J)
      TEMP=S1+SUM1
      SUM1=(S1-TEMP)+SUM1
      S1=TEMP
      SUM4=SUM4+Y(J)
      TEMP=S4+SUM4
      SUM4=(S4-TEMP)+SUM4
      S4=TEMP
      SUM5=SUM5+Y(J)**2
      TEMP=S5+SUM5
      SUM5=(S5-TEMP)+SUM5
      S5=TEMP
   55 CONTINUE
      GO TO (57,100),K
   57 GO TO (58,59),KK
   58 CONTINUE
      GO TO 15
   59 CONTINUE
      GO TO 17
   98 END FILE 1
   99 REWIND 1
  100 AN=I
      K=2
      AK2=(S1-(S2*S4)/AN)/(S3-(S2**2)/AN)
      AK1=S4/AN-(AK2*(S2/AN))
      KRSUM=0
      KASUM=0
      DO 41 J=1,L
      ESTY(J)=AK1+AK2*X(J)
      ESTU(J)=ESTY(J)-CK2*X(J)
      IF (H(J).EQ.0) GO TO 72
   70 IF (ESTU(J).GE.0.0) GO TO 73
   71 KRCT(J)=0
      GO TO 74
   72 IF (ESTU(J).LT.0.0) GO TO 73
```

```
      GO TO 71
   73 KRCT(J)=1
   74 KRSUM=KRSUM+KRCT(J)
      IF(M(J).EQ.0) GO TO 82
      IF (U(J).GE.1.0)  GO  TO 83
   81 KACT(J)=0
      GO TO 84
   82 IF (U(J).LT.1.0) GO TO 83
      GO TO 81
   83 KACT(J)=1
   84 KASUM=KASUM+KACT(J)
   41 CONTINUE
      PCT=100*KRSUM/L
      PCTA=100*KASUM/L
   47 DO 60 J=1,L
      IF(LINE-55)50,50,48
   48 PRINT 9003,HOL
 9003 FORMAT(1H110X,12A6/1H06X,2HID2X,3HNO.4X,4HXF/X4X,4HYF/Y7X,1HA7X,
     15HKY/KX6X,3HX/Y5X,7HL KY/KX5X,5HL X/Y4X,8HEL KY/KX4X,1HM4X,
     25HEST A5X,2HYR2X,2HM11X,4H PE ,4H PA )
      PRINT 9020,ISSI,ISI
 9020 FORMAT (17H CODE OF MODEL IS ,I2,17H PRINTOUT CODE IS ,I2)
      LINE=0
   50 IF(ISI)  53,60,53
   53 PRINT 9004,ID(J),J,A(J),B(J),U(J),DA(J),ARG(J),Y(J),X(J),
     1ESTY(J),M(J),ESTU(J),IDYR(J),M3(J),KRCT(J),KACT(J)
      LINE=LINE+1
 9004 FORMAT (5X,I4,I5,2F8.3,3F10.3,3F11.5,4X,I1,F10.5,2X,4I4)
   60 CONTINUE
  103 SYX=SQRTF((S5-S4**2/AN-(S1-S2*S4/AN)**2/(S3-S2**2/AN))/(AN-2.))
      PRINT 9005,AK1,AK2,S2,S4,S1,S3,S5,SYX,CK2,PCT,PCTA,I
 9005 FORMAT      (///4X16HK1 (INTERCEPT) =F8.3,5X12HK2 (SLOPE) =F8.3/4X3H
     1SX=F11.5,4X3HSY=F11.5,4X4HSXY=F11.5/4X4HSX2=F10.5,4X4HSY2=F10.5,
     24X5HSDYX=F8.5,10X,5H CK2=,F6.2,10X,5H PCT=,F6.1,6X,6H PCTA=,F6.1,
     34X,3H N=,I4)
      TALF=0.674
      GO TO (142,142,143),ISSI
  142 Z(1)=-2.5
      DO 145  J=2,10
      Z(J)=Z(J-1)+0.5
      GO TO 144
  143 Z(1)=-0.25
      DO 145JJ=2,10
      J=JJ
      Z(J)=Z(J-1)+0.25
  144 W(J)=EXPF(Z(J))
      ETY(J)=AK1+AK2*Z(J)
       CT(J)=SQRTF(1+1/AN+(Z(J)-S2/AN)**2/(S3-S2**2/AN))
      PU(J)=ETY(J)+TALF*SYX*CT(J)
      PL(J)=ETY(J)-TALF*SYX*CT(J)
      ETU(J)=ETY(J)-CK2*Z(J)
      T(J)=ETU(J)/(SYX*CT(J))
      AU(J)=PU(J)-CK2*Z(J)
```

```
      AL(J)=PL(J)-CK2*Z(J)
      IF(J-2 ) 146,146,147
  146 PRINT 9008
 9008 FORMAT(//2X,3HX/Y5X,5HI X/Y1X,8HEL KY/KY4X,2HUL6X,
     12HLL6X,1HT7X,1HA7X,2HAU6X,2HAL)
  147 PRINT 9009,W(J),Z(J),ETY(J),PU(J),PL(J),CT(J),T(J),ETU(J),
     1AU(J),AL(J)
 9009 FORMAT(F7.3,9F8.4)
  145 CONTINUE
      IF(CK2+1.0) 195,195,105
  105 CK2=CK2-0.25
      IF (ABS(CK2)-.001) 905,905,106
  905 CK2=0.01
      GO TO 108
  106 IF (ABS(CK2+0.24)-0.021) 107,107,108
  107 CK2=-0.25
  108 N=1
      L=I
      SUM1=0.0
      SUM4=0.0
      SUM5=0.0
      S1=0.0
      S4=0.0
      S5=0.0
      GO TO 45
  195 KN=KN+1
      IF (ISSN(KN)) 196,196,197
  196 STOP
  197 ISSI=ISSN(KN)
      ISI=ISPN(KN)
      REWIND 1
      K=1
      KK=2
      CK2=1.0
      LINE= 56
      ASSIGN 100 TO KEOF
      SUM1=0.0
      SUM2=0.0
      SUM3=0.0
      SUM4=0.0
      SUM5=0.0
      S1=0.0
      S2=0.0
      S3=0.0
      S4=0.0
      S5=0.0
      GO TO 17
      END
```

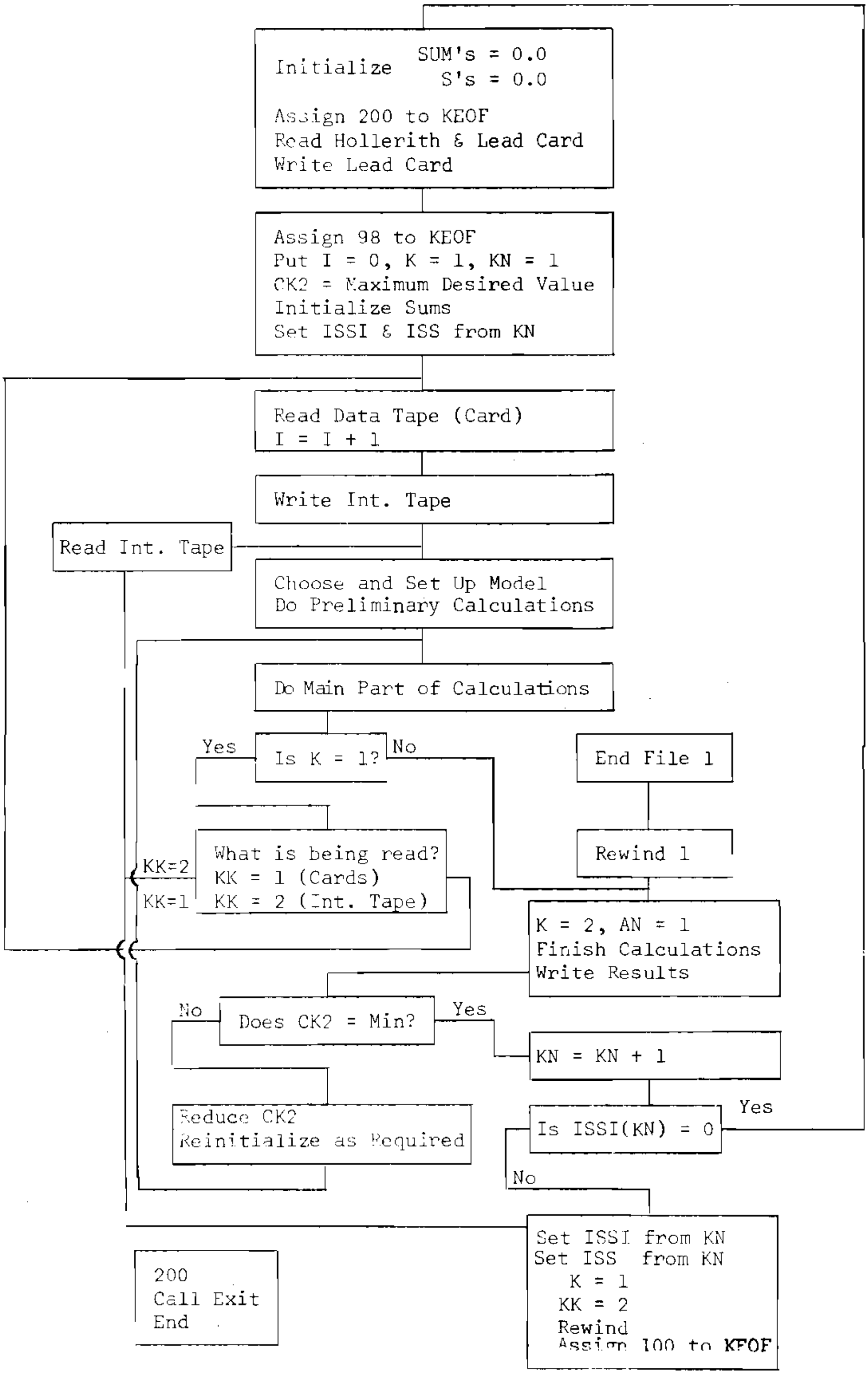

Figure C-1. Flow Diagram for Computer Program

Table C-1. Sample Computer Output for Control Model

ID	XF/X	YF/Y	A	KY/KX	X/Y	L KY/KX	L X/Y	E1 KY/KX	EST A	YR	M1
492	0.992	0.812	0.049	0.109	1.484	-0.96363	0.17154	0.99260	0.67119	620	1
503	0.750	0.750	1.000	1.000	1.000	0.00000	0.00000	0.54325	0.73706	621	1
511	0.923	0.619	0.240	0.368	1.238	-0.43457	0.09275	0.75256	0.70068	622	1
512	0.846	0.882	1.283	0.750	0.750	-0.12494	-0.11651	0.36076	0.78544	622	2
•	•	•	•	•	•	•	•	•	•	•	•
•	•	•	•	•	•	•	•	•	•	•	•
•	•	•	•	•	•	•	•	•	•	•	•
•	•	•	•	•	•	•	•	•	•	•	•
5951	0.937	0.917	0.758	0.526	0.833	-0.27861	-0.07918	0.41131	0.76960	904	1
5972	0.870	0.890	1.166	0.945	0.900	-0.02474	-0.04576	0.46257	0.75569	904	1
5981	0.883	0.781	0.566	0.270	0.690	-0.56902	-0.16085	0.30870	0.80468	904	1
5982	0.464	0.312	0.869	10.649	3.500	1.02732	0.54407	3.67492	0.54772	904	1
5991	0.765	0.776	1.042	0.496	0.690	-0.30485	-0.16137	0.30814	0.80490	905	1
5992	0.869	0.771	0.602	0.617	1.013	-0.20941	0.00557	0.55398	0.73482	905	1

K1 (INTERCEPT) = -0.265	PCTA = 78.3	SX2 = 40.35258
K2 (SLOPE) = 1.526	SX = 17.64350	SY2 = 51.59172
CK2 = 2.00	SY = 08.66592	SDYX = 0.63595
PCT = 97.5	SXY = 82.97916	

Table C-2. Sample Computer Output for Initial Force Model

ID	XF/X	YF/Y	A	KY/KX	X/Y	L KY/KX	L X/Y	E1 KY/KX	M	EST A	YR	M1
551	0.861	0.765	0.623	0.698	1.059	-0.15620	0.02482	0.96589	0	0.92820	631	2
552	0.970	0.889	0.284	0.425	1.222	-0.37178	0.08715	1.18201	0	0.88953	632	2
553	0.983	0.957	0.389	0.661	1.304	-0.17973	0.11539	1.29526	0	0.87254	632	1
561	0.800	0.800	1.000	1.562	1.250	0.19382	0.09691	1.21998	1	0.88362	632	2
•	•	•	•	•	•	•	•	•	•	•	•	•
•	•	•	•	•	•	•	•	•	•	•	•	•
•	•	•	•	•	•	•	•	•	•	•	•	•
•	•	•	•	•	•	•	•	•	•	•	•	•
4651	0.911	0.838	0.570	0.843	1.216	-0.07410	0.08501	1.17384	1	0.89083	813	1
4723	0.929	0.875	0.588	1.800	1.750	0.25527	0.24304	1.95864	0	0.79972	814	1
4983	0.864	0.789	0.672	2.985	2.108	0.47492	0.32389	2.54517	0	0.75677	831	2
5231	0.901	0.882	0.846	2.134	1.588	0.32916	0.20091	1.70878	0	0.82305	862	1
5302	0.958	0.960	1.056	28.566	5.200	1.45585	0.71600	9.06602	0	0.57904	863	1
5602	0.970	0.967	0.902	1.113	1.111	0.04649	0.04576	1.03367	0	0.91502	870	1

K1 (INTERCEPT) = -0.050 PCTA = 78.9 SX2 = 39.44024
K2 (SLOPE) = 1.407 SX = 291.69314 SY2 = 45.61506
CK2 = 2.00 SY = 356.36224 SDYX = 0.55143
PCT = 66.0 SXY = 181.53958

Table C-3. Sample Computer Output for Attacker-Defender Model

ID	XF/X	YF/Y	A	KY/KX	X/Y	L KY/KX	L X/Y	EL KY/KX	M	EST A	YR	M1
492	0.992	0.812	0.049	0.109	1.484	-0.96363	0.17154	0.21618	0	0.74661	620	1
503	0.750	0.750	1.000	1.000	1.000	0.00000	0.00000	-0.00700	0	0.98401	621	1
511	0.923	0.619	0.240	0.368	1.238	-0.43457	0.09275	0.11367	0	0.84755	622	1
512	0.882	0.846	0.780	1.333	1.308	0.12494	0.11651	0.14457	1	0.81576	622	2
•	•	•	•	•	•	•	•	•	•	•	•	•
•	•	•	•	•	•	•	•	•	•	•	•	•
•	•	•	•	•	•	•	•	•	•	•	•	•
•	•	•	•	•	•	•	•	•	•	•	•	•
5951	0.937	0.917	0.758	0.526	0.833	-0.27861	-0.07918	-0.11001	0	1.11776	904	1
5972	0.870	0.890	1.166	0.945	0.900	-0.02474	-0.04576	-0.06653	0	1.05922	904	1
5981	0.883	0.781	0.566	0.270	0.690	-0.56902	-0.16085	-0.21627	0	1.27478	904	1
5982	0.464	0.312	0.869	10.649	3.500	1.02732	0.54407	0.70083	0	0.40992	904	1
5991	0.765	0.776	1.042	0.496	0.690	-0.30485	-0.16137	-0.21694	0	1.27584	905	1
5992	0.869	0.711	0.602	0.617	1.013	-0.20941	0.00557	0.00024	0	0.97523	905	1

K1 (INTERCEPT) = -0.007	PCTA = 79.1	SX2 = 139.90105
K2 (SLOPE) = 1.301	SX = 43.42448	SY2 = 448.00352
CK2 = 2.00	SY = 49.21796	SDYX = 0.64092
PCT = 64.9	SXY = 181.75415	

Computer Program for Sensitivity Analysis

This FORTRAN program was developed to study the effects of varying the effectiveness coefficients in Lanchester's square law and Lanchester's linear law. The solution to both the square and linear laws are programmed to give printout of the remaining strength of both sides in steps of t = 0.1. K_{12} and K_{21} are varied in five equal steps. The size of these steps along with the initial values of K_{12} and K_{21} are required as input data. A format for the input data follows the FORTRAN listing.

FORTRAN Listing

```
1 FORMAT(E10.3,E10.3,F10.0,F10.0,E10.3,E10.3,110)
2 FORMAT(///12X3HX10,7X3HX20,7X3HK12,9X3HK21,10X3HCAT)
3 FORMAT (/6XF10.0,F10.0,E12.3,E12.3,110)
4 FORMAT (/11X1HT,7X3HX1L,7X3HX2L,7X3HX1S,7X3HX2S/)
5 FORMAT (8XF5.1,110,110,110,110)
  DIMENSION YK12(5),YK21(5),T(30)
  N=1
8 READ 1,YK12,YK21,X10,X20,A,B,C
  YK12(1)=YK12
  YK21(1)=YK21
  T(1)=0.0
  DO 30 K=1,5
  DO 20 J=1,5
  DO 10 1=1,3C
  Y=YK21(J)*X20-YK12(K)*X10
  IX1L=X10*Y/(YK21(J)*X20*EXP(Y*T(1))-YK12(K)*X10)
  IX2L=X20*(-Y)/(YK12(K)*X10*EXP(-Y*T(1))-YK21(J)*X20)
  U=SQRT(YK12(K)*YK21(J)*X10*X20)
  V=SQRT((YK21(J)*X10)/(YK12(K)*X20))
  W=SQRT((YK12(K)*X20)/(YK21(J)*X10))
  R=EXP(U*T(1))
  S=1.0/R
  IX1S=X10*(R+S)/2.0-X20*V*(R-S)/2.0
  IX2S=X20*(R+S)/2.0-X10*W*(R-S)/2.0
  IF(I-1) 6,6,7
6 PUNCH 2
  PUNCH 3,X10,X20,YK12(K),YK21(J),C
  PUNCH 4
7 PUNCH 5,T(I),IX1L,IX2L,IX1S,IX2S
  T(I+1)=T(1)+0.1
```

```
10 CONTINUE
   YK21(J+1)=YK21(J)+B
20 CONTINUE
   YK12(K+1)=YK12(K)+A
30 CONTINUE
   N=N+1
   IF(N-5) 8,40,40
40 STOP
   END
```

<u>Input Format</u>

A separate data card must be prepared for each battle to be considered. From one to four data cards can be stacked at the end of the deck. Each data card must be prepared according to the following format:

1. Columns 1-10. Effectiveness coefficient of the superior force expressed in E conversion format right justified on Column 10.

2. Columns 11-20. Effectiveness coefficient of the inferior force expressed in E conversion format right justified on Column 20.

3. Columns 21-30. Initial strength of the superior force right justified on Column 30.

4. Columns 31-40. Initial strength of the inferior force right justified on Column 40.

5. Columns 41-50. Incremental step size of the superior force effectiveness coefficient right justified on Column 50.

6. Columns 51-60. Incremental step size of the inferior force effectiveness coefficient right justified on Column 70.

7. Columns 61-70. Battle identification number (must be an integer) right justified on Column 70.

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VITA

William Anton Schmieman was born July 6, 1921, the son of Carl G. and Gertrude (Geiger) Schmieman. He has three brothers, Gustav, Thomas, and Robert, and an older sister, Mary. He lived within 50 miles of Fort Wayne, Indiana during his youth and graduated from Central High School in Fort Wayne with the Class of 1939.

He took the college preparatory course in high school and had planned to enter Stanford University in September, 1939; however, his father died unexpectedly and he took a position with the General Electric Company, where he was employed until entering the Army.

He enlisted in the Army Reserve in May, 1942, and went on active duty with the Army Air Corps in October, 1942. He became an airborne radio operator on the medium bombardment aircraft, B-26, the Martin Marauder, and flew 65 combat missions in the Mediterranean Theatre of Operations with the 319th Bombardment Group. After his return to the United States, he was assigned to Shaw Field, Sumter, S. C., as NCO in charge of Technical Training and taught radio operation and procedure to pilots and members of the First Air Force, First Fighter Command. In September of 1945, he was honorably discharged. He returned to Fort Wayne, Indiana and resumed his work with the General Electric Company.

In November of 1945, he married the former Miss Fern Thomas of Portland, Indiana. In the winter quarter of 1945, he enrolled as a full-time engineering student at Indiana Technical College and studied electrical engineering. At the beginning of the 1947 fall quarter, he transferred to Ball State University, Muncie, Indiana as a physics-mathematics major and received his B.S. degree in December of 1948.

He accepted a civil service position as a Radio Engineer at Wright-Patterson Air Force Base in January of 1949 and has been continuously employed by the Federal Civil Service in a professional capacity since that time. During his Civil Service career, he has taken graduate work in mathematics and theoretical physics at the Ohio State University, University of Florida, and the Florida State University. In Decmeber of 1953 he accepted a position with the Technical Analysis Office of the Air Force Armament Center at Eglin Air Force Base, Florida and has been in weapons test design and analysis work since that time.

In February, 1961, he was chosen as one of five civil service employees at Eglin Air Force Base to receive a full calendar year's training at the doctorate level in a university and in the field of his choice.

He was accepted by the School of Industrial Engineering, Georgia Institute of Technology and commenced work toward the doctorate degree in September of 1961. He fulfilled the requirements and was admitted to candidacy for the degree, Doctor of Philosophy, in the School of Industrial Engineering, on June 26, 1963.

Upon finishing the academic requirements for this degree at the Georgia Institute of Technology, he returned to Eglin AFB to accept the position of Technical Director, Directorate of Technical Analysis, Deputy for Effectiveness Test, Air Proving Ground Center, Eglin Air

Force Base, Florida. In January, 1966, Mr. Schmieman accepted the position of Chief, Operations Analysis Office, National Range Division at Patrick Air Force Base, Florida, and he is currently serving in this capacity.